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WITT CLASSES OF INTEGRAL REPRESENTATIONS OF AN ABELIAN p-GROUP

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1. **Introduction.** For a Dedekind domain, R, the orthogonal and symplectic representations of a finite group, π, on finitely-generated projective inner-product modules over R admit a Witt equivalence relation, and the resulting equivalence classes form a commutative algebra, $\mathcal{W}_*(R, \pi)$, over the Witt ring of R. This concept has received considerable attention recently [2], [3], [4]. Our interest is motivated by the fact that $\mathcal{W}_*(Z, \pi)$ is so very specifically related to the bordism classification of smooth, orientation preserving actions of π on closed even-dimensional manifolds. We shall discuss

(1.1) THEOREM. *If, for p an odd prime, π is an abelian p-group then $\mathcal{W}_*(Z, \pi)$ contains no torsion.*

A corollary of (1.1) is that for an action (π, M^{2k}) of such a group on a closed oriented manifold, the Atiyah-Singer-Segal G-signature theorem [1] determines the integral Witt class of $(\pi, H^*(M; Z)/\mathrm{tor})$ uniquely. The present techniques may also be applied to determine $\mathcal{W}_*(Z, \pi)$ for an abelian 2-group, however torsion is present always. Thus for an orientation preserving action (π, M^{2k}) of an abelian 2-group, a torsion valued invariant, as well as the multisignature, must be computed.

By rough analogy with [5, IV, (3.3)] there is

(1.2) LEMMA. *For any p-group*

$$\mathcal{W}_2(Z, \pi) \simeq \mathcal{W}_2(Z(1/p), \pi),$$

and there is a split short exact sequence

$$0 \longrightarrow \mathcal{W}_0(Z, \pi) \longrightarrow \mathcal{W}_0(Z(1/p), \pi) \longrightarrow \mathcal{W}(Z_p) \longrightarrow 0.$$

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We use the subscripts 0 and 2 respectively to denote orthogonal and symplectic representations.

2. **Cyclic p-groups.** From this point we restrict our attention to odd primes. For $n \geqslant 0$ we denote by $Q(\lambda)$ the p^{n+1}-cyclotomic extension of the rationals. In the ring of algebraic integers, $Z(\lambda)$, there is the multiplicative subset, S, generated by the rational prime, p, and $S^{-1}Z(\lambda) = D \subset Q(\lambda)$ is a Dedekind domain invariant under complex conjugation. We may thus speak of the Witt ring of Hermitian inner-product modules over D, $\mathcal{H}_0(D)$, and by introducing skew-Hermitian inner-products there is $\mathcal{H}_2(D)$ and hence an algebra, $\mathcal{H}_*(D)$.

(2.1) LEMMA. *For $n \geqslant 0$ there is an additive isomorphism*

$$\mathcal{W}_*(Z(1/p), Z_{p^{n+1}}) \simeq \mathcal{W}_*(Z(1/p), Z_{p^n}) \oplus \mathcal{H}_*(D).$$

Very briefly, we consider a $(Z_{p^{n+1}}, V)$ where V is an inner-product module over $Z(1/p)$ and choose a generator $T \in Z_{p^{n+1}}$. With $\tau = T^{p^n}$, we introduce into V a selfadjoint projection operator

$$\sum v = (v + \tau(v) + \cdots + \tau^{p-1}(v))/p.$$

This yields an orthogonal decomposition $V = I \oplus I^{\perp}$ into the image, I, of Σ and the kernel, $I^{\perp}$. On I the subgroup generated by τ acts trivially, so we may replace $Z_{p^{n+1}}$ by the quotient group Z_{p^n}. Now D is the quotient of the group ring $Z(1/p)(Z_{p^{n+1}})$ by the principal ideal which $1 + \tau + \cdots + \tau^{p-1}$ generates. In this fashion $I^{\perp}$ becomes a projective D-module. The (skew-)Hermitian inner-product on $I^{\perp}$ is $[v, w] = \Sigma_j (v, T^j w)\lambda^j$.

At this point standard algebraic number theory intervenes in proving

(2.2) LEMMA. *The Hermitian Witt ring $\mathcal{H}_0(D)$ has no torsion.*

Denoting by $Q(\lambda + \lambda^{-1})$ the subfield of real elements in $Q(\lambda)$, we may paraphrase the discussion in [5, IV, §4] to show that $v \in Q(\lambda + \lambda^{-1})^*$ can, up to mulitplication by a Hermitian square, be realized as the discriminant of a Hermitian inner-product module over D with even rank if and only if $LL^- = vD$ for some fractional D-ideal $L \subset Q(\lambda)$. The key lemma then is

(2.3) LEMMA. *If $LL^- = vD$ then v is a Hermitian square if and only if it is positive in every ordering of $Q(\lambda + \lambda^{-1})$.*

The lemma depends on the fact that only the rational prime p ramifies in $Q(\lambda)$. It then proceeds from a combination of the local norm index theorem for units [**6**, IX, p. 187] with the reciprocity law for Hilbert symbols in the number field $Q(\lambda + \lambda^{-1})$. As a consequence of Landherr [**5**, p. 118, Example 4], this lemma eliminates torsion in $H_0(D)$. It is possible to determine $H_0(D)$ completely. It is necessary to produce elements $v \in Q(\lambda + \lambda^{-1})^*$ with arbitrarily prescribed signs and satisfying $LL^{-1} = vD$. This cannot be accomplished, in general, by only selecting units in $D^* \cap Q(\lambda + \lambda^{-1})^*$, and the argument involves a careful analysis of the role of the homology groups of Z_2 acting on the ideal class group of D via conjugation of fractional D-ideals.

Beginning with $n = 0$, Lemmas (2.2), (2.1) and (1.2) are combined inductively to yield (1.1) for the cyclic p-groups. Since D^* contains an imaginary unit, $H_0(D) \simeq H_2(D)$ additively so there is no special problem in handling the symplectic case.

3. **The general case.** We express π as a direct sum $\pi = \pi_1 \oplus Z_{p^{n+1}}$ with n as large as possible. Now proceeding as in (2.1) we split $W_*(Z(1/p), \pi)$ into a direct sum $W_*(Z(1/p), \pi_1 \oplus Z_{p^n}) \oplus H_*(D, \pi_1)$. From our choice of n we may identify the character group π_1^* with Hom (π_1, D^*). It follows readily that $H_*(D, \pi_1) \simeq Z(\pi_1^*) \otimes H_*(D)$. Combining (2.1) and (1.2) with this observation, (1.1) is established.

For a specific example if p is odd, the multisignature is an isomorphism of $W_*(Z, Z_p)$ onto the subring of the group ring $Z(Z_p)$ consisting of those elements which can be expressed in the form

$$m_0 \cdot e + \sum_1^k m_j(\tau^j + \tau^{-j}) + \sum_1^k n_j(\tau^j - \tau^{-j}),$$

where $k = (p - 1)/2$ and the integral coefficients satisfy $m_1 = m_2 = \cdots = m_k \pmod 2$ and $n_1 = n_2 = \cdots = n_k \pmod 2$.

ADDED IN PROOF. Theorem (1.1) can be proven for general p-groups π, for example by the induction techniques of Dress [**2**].

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