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THE COMPUTATION OF SURGERY GROUPS OF ODD TORSION GROUPS

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The purpose of this note is to describe some of the main results pertaining to the computation of surgery groups of finite groups found in a joint paper [7] with W. Scharlau, one book [1] and five papers [2]-[6] of the author. Mention of the results is made in [10]. I shall indicate the source of each result. Throughout this note π denotes a group. Let us begin with the results for surgery groups of odd torsion groups. Results closely related to the first three theorems have been announced in Wall [12].

THEOREM 1 [2]. *If π is an odd torsion group then the surgery obstruction groups*

$$L^{s,h}_{2n+1}(\pi) = 0.$$

Let r_∞ denote the number (infinite if π is infinite) of irreducible real representations of π.

THEOREM 2 [3]. *If π is an odd torsion group then the surgery obstruction groups*

$$\begin{aligned} L^{s}_{2n}(\pi) &= Z^{r_\infty} && \text{if } n \equiv 0 \bmod 2,\\ &= Z^{r_\infty-1} \oplus Z_2 && \text{if } n \equiv 1 \bmod 2, \end{aligned}$$

and in the latter case the nontrivial element of Z_2 is represented by the based quadratic form $(Z\pi\oplus Z\pi, \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right))$.

Let $Z\pi$ and $Q\pi$ be the integral and rational group rings of π. Let $K_0(Z\pi, Q\pi)$ be the relative group in the exact sequence of a localization [9, IX, §6] and let $\tilde{K}_0(Z\pi)=K_0(Z\pi)/[Z\pi]$ be the projective class group of $Z\pi$. $K_0(Z\pi, Q\pi)$ is generated by pairs (M, N) of finitely-generated projective $Z\pi$-lattices on a free $Q\pi$-module and if $M^*=\operatorname{Hom}_{Z\pi}(M, Z\pi)$, then $K_0(Z\pi, Q\pi)$ has a Z_2-action defined by $(M, N)\mapsto -(M^*, N^*)$, and $\tilde{K}_0(Z\pi)$ a Z_2-action defined by $M\mapsto -M^*$. Let $H^0(K_0(Z\pi, Q\pi))$ be the zeroth cohomology group of the Z_2-action on $K_0(Z\pi, Q\pi)$ and let

$$H(\pi) = \operatorname{coker} H^0(K_0(Z\pi, Q\pi)) \to \tilde{K}_0(Z\pi).$$

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In [6] I show that when π is finite abelian, $K_0(Z\pi, Q\pi) \simeq I(Z\pi)$ (=group of invertible fractional $Z\pi$-ideals), $\tilde{K}_0(Z\pi) \simeq \mathrm{Cl}(Z\pi)$ $(=I(Z\pi)/$principal fractional ideals), and the Z_2-actions on $K_0(Z\pi, Q\pi)$ and $\tilde{K}_0(Z\pi)$ correspond to the natural involutions on $I(Z\pi)$ and $\mathrm{Cl}(Z\pi)$ given by the involution on $Q\pi$.

THEOREM 3 [3]. *If π is an odd torsion group then*

$$\begin{aligned} L^h_{2n}(\pi) &= Z^{r_\infty} \oplus H(\pi) && \text{if } n \equiv 0 \bmod 2, \\ &= Z^{r_\infty - 1} \oplus Z_2 \oplus H(\pi) && \text{if } n \equiv 1 \bmod 2. \end{aligned}$$

The nontrivial element of Z_2 is represented by the quadratic form $(Z\pi \oplus Z\pi, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right))$ and the elements of $H(\pi)$ correspond to the classes of hyperbolic modules $H(M)$ such that M is finitely-generated projective and $M \oplus M^*$ is free.

Let A be a ring with involution and let $\lambda \in$ center A such that $\lambda\bar{\lambda}=1$. Let $KU_0^\lambda(A)$ $(=KQ_0^\lambda(A, \max)$, see [1, §1B]) be the Grothendieck group of nonsingular even λ-hermitian forms on finitely-generated projective A-modules and let $W_0(A)=KU_0(A)/$hyperbolic modules.

THEOREM 4 [3]. *Let*
F number field with involution,
E fixed field of the involution,
S ring of integers in F,
λ $\lambda \in S$ such that $\lambda\bar{\lambda}=1$,
π finite odd order or finite abelian group.
Give $S\pi$ the involution which sends each element of π to its inverse and agrees with the involution on S.

(i) *Assume F has nontrivial involution, E has an undecomposed real prime, and F/E is totally unramified. Then $W_0(S\pi)$ is torsion free of the same (finite) rank as $W_0(F\pi)$.*

(ii) *Assume F has trivial involution, F is real, and each rational prime $|$ $[\pi:1]$ is inert. Assume either $\lambda=-1$ or the class number of F is odd. Then $W_0(S\pi)$ is torsion free of the same (finite) rank as $W_0(F\pi)$.*

Let $KQ_0^\lambda(A)$ $(=KQ_0^\lambda(A, \min)$, see [1, §1B]) be the Grothendieck group of nonsingular quadratic forms on finitely-generated projective A-modules.

THEOREM 5 [3]. *If π is a finite odd order group then*

$$\begin{aligned} KU_0^\lambda(Z\pi) &= Z^{r_\infty+1} \oplus H(\pi) && \text{if } \lambda = 1, \\ &= Z^{r_\infty} \oplus H(\pi) && \text{if } \lambda = -1. \\ KQ_0^\lambda(Z\pi) &= Z^{r_\infty+1} \oplus H(\pi) && \text{if } \lambda = 1, \\ &= Z^{r_\infty} \oplus Z_2 \oplus H(\pi) && \text{if } \lambda = -1. \end{aligned}$$

There is a general procedure described in [**1**, §§1B, 3] to obtain results on Grothendieck groups of quadratic forms from Grothendieck groups of hermitian forms, and vise versa. We apply this now to the case of group rings. Let $KQ_0^{\pm 1}(Z\pi)_{\text{based, proj}}$ (resp. $KU_0^{\pm 1}(Z\pi)_{\text{based, proj}}$) be the Grothendieck groups of nonsingular quadratic (resp ± 1-hermitian) forms on finitely-generated based or projective $Z\pi$-modules. Let $WQ_0^{\pm 1}(Z\pi)_{\text{based, proj}}$ (resp. $W_0^{\pm 1}(Z\pi)_{\text{based, proj}}$) be $KQ_0^{\pm 1}(Z\pi)_{\text{based, proj}}$ (resp. $KU_0^{\pm 1}(Z\pi)_{\text{based, proj}}$) modulo hyperbolic modules on based or projective modules.

THEOREM 6 [**1**]. *No assumption that π be finite is made.*

(a)
$$KQ_0^{+1}(Z\pi)_{\text{based,proj}} \xrightarrow{\cong} KU_0^{+1}(Z\pi)_{\text{based, proj}},$$
$$WQ_0^{+1}(Z\pi)_{\text{based,proj}} \xrightarrow{\cong} W_0^{+1}(Z\pi)_{\text{based,proj}}.$$

(b) *Assume that the elements of exponent 2 in π generate a nilpotent subgroup (equivalently the 2-torsion elements generate a 2-group). Then the sequences below are split exact*

$$0 \to Z_2 \to KQ_0^{-1}(Z\pi)_{\text{based,proj}} \to KU_0^{-1}(Z\pi)_{\text{based,proj}} \to 0,$$
$$0 \to Z_2 \to WQ_0^{-1}(Z\pi)_{\text{based,proj}} \to W_0^{-1}(Z\pi)_{\text{based,proj}} \to 0,$$

and in both cases Z_2 is generated by the difference $[Z\pi \oplus Z\pi, \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)] - [H(Z\pi)]$.

Let $KQ_1^{\pm 1}(Z\pi)$ (resp. $KU_1^{\pm 1}(Z\pi)$) be K_1 of the category of nonsingular quadratic (resp. even ± 1-hermitian) forms on finitely-generated projective modules. For the standard matrix definition of these groups see [**1**, §§5, 6]. The next result has been obtained by Bass [**8**] in the case $\lambda = -1$ and by Siu [**11**] in the case $\lambda = 1$ and π cyclic.

THEOREM 7 [**5**]. *Let π be an odd torsion abelian group. Then there are split exact sequences*

$$0 \to Z_2 \to KQ_1^1(Z\pi) = KU_1^1(Z\pi) \xrightarrow{\det} \pm\pi \to 0,$$
$$0 \to Z_4 \to KQ_1^{-1}(Z\pi) \xrightarrow{\det} \pi \to 0,$$
$$KU_1^{-1}(Z\pi) \underset{\cong}{\xrightarrow{\det}} \pi.$$

In the first case Z_2 is generated by the class of the matrix $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ *and in the second case Z_4 by* $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

For a related result in the nonabelian case see [**2**].

Give $\mathrm{Wh}(\pi)$ the involution defined by the conjugate transpose. The next result has been obtained also by Bass [**8**, §4].

THEOREM 8 [**4**]. *If π is a torsion abelian group then the involution on the Whitehead group* $\mathrm{Wh}(\pi)$ *is trivial.*

Let $\hat{H}_0(\mathrm{Wh}(\pi))$ and $\hat{H}^0(\mathrm{Wh}(\pi))$ denote the reduced homology and cohomology groups of the involution on $\mathrm{Wh}(\pi)$. Let r=number of irreducible rational representations of π and let $r_2=Z_2$-rank $(Z_2\otimes\mathrm{Wh}(\pi))$. The case π abelian of the next theorem has been obtained also by Bass [**8**, §4].

THEOREM 9 [**4**]. (a) *If π is a finite odd torsion group then*

$$\hat{H}_0(\mathrm{Wh}(\pi)) = 0, \qquad \hat{H}^0(\mathrm{Wh}(\pi)) = Z_2^{r_\infty - r}.$$

(b) *If π is a finite abelian group then*

$$\hat{H}_0(\mathrm{Wh}(\pi)) = Z_2^{r_2}, \qquad \hat{H}^0(\mathrm{Wh}(\pi)) = Z_2^{r_\infty - r + r_2}.$$

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