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SOME EXAMPLES OF SPHERE BUNDLES OVER SPHERES WHICH ARE LOOP SPACES mod p

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ABSTRACT. In this note we give sufficient conditions that certain sphere bundles over spheres, denoted $B_n(p)$, are of the homotopy type of loop spaces mod p for p an odd prime. The method is to construct a classifying space for the p-profinite completion of $B_n(p)$ by collapsing an Eilenberg-Mac Lane space by the action of a certain finite group.

We say that a space X has some property mod p if the localization of X at p has the property. The problem of determining which spheres are of the homotopy type of loop spaces mod p has been completely solved by Sullivan [9]. It is therefore natural to ask which sphere bundles over spheres are of the homotopy type of loop spaces mod p. In this regard, results of Curtis [2] and Stasheff [7] concerning the question of which sphere bundles over spheres are H-spaces mod p give some negative information. Moreover, in a recent paper [3] we investigated a certain class of sphere bundles over spheres and gave necessary conditions for them to be of the homotopy type of a loop space mod p for p an odd prime. In this note we prove that certain of these bundles satisfying the conditions of [3] are of the homotopy type of a loop space mod p and answer a question posed in [8].

For p an odd prime and n a positive integer, the space $B_n(p)$ is an S^{2n+1}-bundle over $S^{2n+1+2(p-1)}$ classified by the generator of the p-primary part of $\pi_{2n+2(p-1)}(S^{2n+1})$. From [5] we have that $H^*(B_n(p); Z/p)$ is an exterior algebra on generators x and y, where $\deg x=2n+1$, $\deg y=2n+2p-1$ and $\mathcal{P}^1 x=y$. Although few of the $B_n(p)$ are of the homotopy type of a loop space mod p (see [3]), we have the following exceptions.

THEOREM 1. *The space $B_n(p)$ is of the homotopy type of a loop space* mod p *if n and p satisfy any of the following conditions*:

(i) $n=1$; $p=$ *any odd prime*,
(ii) $n=p-2$; $p=$ *any odd prime*,
(iii) $n=7$; $p=17$,
(iv) $n=5$; $p=19$,
(v) $n=19$; $p=41$.

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REMARK. Two cases of Theorem 1 follow immediately from [5]: namely $B_1(3)$ is of the homotopy type of Sp(2)mod 3, and $B_1(5)$ is of the homotopy type of G_2 mod 5.

In order to prove Theorem 1, we must introduce the p-profinite completion of a space as defined in [9]. For precise statements of some of the pertinent theorems, see [4]. If X is a space, let $\hat{X}_p$ denote the p-profinite completion of X; for notational convenience we make the following conventions:

$$L_n(p) = \text{localization of } B_n(p) \text{ at } p.$$
$$C_n(p) = p\text{-profinite completion of } B_n(p).$$

THEOREM 2. *The space $C_n(p)$ is of the homotopy type of a loop space if n and p satisfy any one of the conditions of Theorem* 1.

PROOF OF THEOREM 1. Theorem 1 now follows from Theorem 2 using techniques of [9]. Suppose $C_n(p)$ is a loop space, and let $BC_n(p)$ denote the classifying space. Let W denote the homotopy pull-back in the following diagram:

$$\begin{array}{ccc} W & \longrightarrow & BC_n(p) \\ \downarrow & & \downarrow \\ K(Q, 2n+2) \times K(Q, 2n+2p) & \longrightarrow & K(Q_p, 2n+2) \times K(Q_p, 2n+2p), \end{array}$$

where Q_p denotes the p-adic numbers. Looping the diagram we conclude that $L_n(p) \simeq \Omega W$. Q.E.D.

The proof of Theorem 2 is somewhat involved and so we outline the procedure. Given n and p satisfying one of the conditions, we construct two p-profinitely complete spaces, A and X, together with a map i: $A \to X$. We show $\Omega A \simeq \hat{S}_p^{2n+1}$, $H^*(\Omega X; Z/p) \approx H^*(B_n(p); Z/p)$ as modules over the Steenrod algebra, and $(\Omega i)^*$: $H^{2n+1}(\Omega X; Z/p) \to H^{2n+1}(\Omega A; Z/p)$ is an isomorphism. We conclude that there is a map $f: S^{2n+1} \to \Omega X$ such that $f^*: H^{2n+1}(\Omega X; Z/p) \to H^{2n+1}(S^{2n+1}; Z/p)$ is an isomorphism. From [5] we have the following cell structure for $B_n(p)$:

$$B_n(p) \cong S^{2n+1} \cup_\alpha e^{2n+2p-1} \cup e^{4n+2p}.$$

Since $\mathscr{P}^1$ is nontrivial on $H^*(\Omega X; Z/p)$, we conclude $f\alpha$ is null homotopic. Therefore, by proving $\pi_{4n+2p-1}(\Omega X)$ is trivial, we have shown that f extends to a map $f: B_n(p) \to \Omega X$. By functoriality of $\mathscr{P}^1$ and cup products, this extension induces an isomorphism of mod p cohomology. From [9] or [4] we have that $C_n(p) \simeq \Omega X$.

In this note we give details of the construction only in the case $n=1$. The remaining cases are similar and details can be found in [4]. Let $\hat{Z}_p$ denote the p-adic integers, and let θ be a primitive $(p+1)$st root of unity.

It is easily verified that $\theta+\theta^{-1}$ and $(\theta-\theta^{-1})^2$ are in $\hat{Z}_p$. Let D_{p+1} denote the dihedral group of order $2(p+1)$ in $\mathrm{GL}(2, \hat{Z}_p)$ generated by

$$\frac{1}{2}\begin{pmatrix} \theta + \theta^{-1} & (\theta - \theta^{-1})^2 \\ 1 & \theta + \theta^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let C_2 denote the cyclic group of order 2 in D_{p+1} generated by the second element above. Let $j:\hat{Z}_p\to\hat{Z}_p\times\hat{Z}_p$ be inclusion into the first factor. We proceed as in [**1**]. The natural actions of C_2 and D_{p+1} on $\hat{Z}_p$ and $\hat{Z}_p\times\hat{Z}_p$ induce actions on the Eilenberg-Mac Lane spaces $K(\hat{Z}_p, 2)$ and $K(\hat{Z}_p\times\hat{Z}_p, 2)$. Let ED_{p+1} be an acyclic complex on which D_{p+1} acts freely, and let C_2 and D_{p+1} act on $K(\hat{Z}_p, 2)\times ED_{p+1}$ and $K(\hat{Z}_p\times\hat{Z}_p, 2)\times ED_{p+1}$ by diagonal actions. Let A and X denote the p-profinite completions of the respective orbit spaces. Denote by $i:A\to X$ the map induced by j. From [**1**] we can conclude:

$$\begin{aligned} &H^*(A; Z/p) \approx Z/p[x], \qquad \deg x = 4;\\ &H^*(X; Z/p) \approx Z/p[u, v], \qquad \deg u = 4, \qquad \deg v = 2p + 2;\\ &\qquad i^*(u) = x \quad \text{and} \quad \mathscr{P}^1 u = v \quad (\text{see } [\mathbf{2}]). \end{aligned}$$

If we consider loop spaces we have correspondingly:

$$\begin{aligned} &H^*(\Omega A; Z/p) \approx E(\bar{x}), \qquad \deg \bar{x} = 3;\\ &H^*(\Omega X; Z/p) \approx E(\bar{u}, \bar{v}), \qquad \deg \bar{u} = 3, \qquad \deg \bar{v} = 2p + 1;\\ &\qquad (\Omega i)^*(\bar{u}) = \bar{v} \quad \text{and} \quad \mathscr{P}^1 \bar{u} = \bar{v}. \end{aligned}$$

Since ΩA is a simply-connected, p-profinitely complete space, we have $\Omega A\simeq\hat{S}^3_p$. Therefore we get a map $f:S^3\to\Omega X$ such that $f^*(\bar{u})\neq 0$. From the above remarks, to extend f to $B_1(p)$ we need only show $\pi_{2p+3}(\Omega X)=0$.

Consider the diagram:

$$\begin{array}{ccc} \Omega X & & \Omega X \\ | & & | \\ E & \longrightarrow & \Lambda X \\ | & & | \\ A & \longrightarrow & X. \end{array}$$

The pull-back E is a simply-connected p-profinitely complete space. Moreover, we can compute $H^*(E; Z/p)$ from the Eilenberg-Moore spectral sequence, which collapses [**6**] and gives $H^*(E; Z/p)$ as an exterior algebra on a generator of degree $2p+1$. We conclude $E\simeq\hat{S}^{2p+1}_p$. Since $\pi_{2p+4}(A)\approx\pi_{2p+3}(\hat{S}^3_p)=0$ and $\pi_{2p+3}(\hat{S}^{2p+1}_p)=0$, we have $\pi_{2p+3}(\Omega X)=0$.

Therefore, f extends to a map $f:B_1(p)\to\Omega X$ which induces an isomorphism on mod p cohomology. We conclude that $C_1(p)\simeq\Omega X$.

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