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ADDITIVE COMMUTATORS BETWEEN 2×2 INTEGRAL MATRIX REPRESENTATIONS OF ORDERS IN IDENTICAL OR DIFFERENT QUADRATIC NUMBER FIELDS

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The following theorem holds:

THEOREM 1. *Let A, B be two integral 2×2 matrices. Let the characteristic roots of A be α, α' and let the characteristic roots of B be β, β', all assumed irrational. Then the determinant of*

$$(*)\qquad L = AB - BA$$

is a negative norm in both $Q(\alpha)$, $Q(\beta)$.

REMARK. The proof of this theorem gives an algorithmic procedure for expressing an integer as a norm in a quadratic field.

PROOF. There exists[2] an integral matrix S with the property that $S^{-1}AS$ is the companion matrix

$$\begin{pmatrix} 0 & 1 \\ -\det A & \operatorname{tr} A \end{pmatrix}$$

of A. Since the companion matrix has the characteristic vectors $(1, \alpha)'$, $(1, \alpha')'$ the matrix $T=\left(\begin{smallmatrix} 1 & 1 \\ \alpha & \alpha' \end{smallmatrix}\right)$ has the property that $T^{-1}S^{-1}AST=\left(\begin{smallmatrix} \alpha & \\ & \alpha' \end{smallmatrix}\right)$. Apply then the same similarity also to B and to L, i.e. to $(*)$. Let the outcome of this be denoted by

$$(**)\qquad \begin{pmatrix} \alpha & \\ & \alpha' \end{pmatrix} B^{(\alpha)} - B^{(\alpha)} \begin{pmatrix} \alpha & \\ & \alpha' \end{pmatrix} = L^{(\alpha)} = \begin{pmatrix} 0 & l_2 \\ l_3 & 0 \end{pmatrix};$$

then l_2, l_3 are elements in $Q(\alpha)$.

Apply the similarity defined by T^{-1} to $L^{(\alpha)}$. The result must be rational. A straightforward computation using the fact that $\alpha, \alpha' = -\frac{1}{2}(\operatorname{tr} A \pm \sqrt{m})$, with $m = (\operatorname{tr} A^2 - 4 \det A)$, shows that

$$\begin{pmatrix} 1 & 1 \\ \alpha & \alpha' \end{pmatrix} \begin{pmatrix} 0 & l_2 \\ l_3 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & -1 \\ -\alpha & 1 \end{pmatrix} \frac{1}{\alpha' - \alpha} = -\frac{1}{\sqrt{m}} \begin{pmatrix} \alpha' l_3 - \alpha l_2 & l_2 - l_3 \\ \alpha'^2 l_3 - \alpha^2 l_2 & -\alpha' l_3 + \alpha l_2 \end{pmatrix}.$$

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[2] For further information in the number theoretic case on this see [1].

This implies

(1) $$l_2 - l_3 = r_1\sqrt{m}, \quad \text{with } r_1 \text{ rational},$$

$$-m^{-1/2}[\tfrac{1}{2}(\operatorname{tr} A - \sqrt{m})l_3 - \tfrac{1}{2}(\operatorname{tr} A + \sqrt{m})l_2]$$

(2) $$= -m^{-1/2}[\tfrac{1}{2} \operatorname{tr} A(l_3 - l_2) - \tfrac{1}{2}\sqrt{m}(l_3 + l_2)]$$

$$= \text{rational}.$$

In virtue of (1) we obtain

(3) $$l_2 + l_3 = r_2, \quad \text{with } r_2 \text{ rational}.$$

From (1), (3) follows

$$l_2 = \tfrac{1}{2}(r_2 + r_1\sqrt{m}), \qquad l_3 = \tfrac{1}{2}(r_2 - r_1\sqrt{m}).$$

Hence l_2, l_3 are conjugate elements in $Q(\alpha)$. Since

$$\det\begin{pmatrix} 0 & l_2 \\ l_3 & 0 \end{pmatrix} = -l_2 l_3 = \det L,$$

the theorem follows if it is further observed that $AB-BA=-(BA-AB)$ and that $\det(AB-BA)=\det(BA-AB)$.

THEOREM 2. *Let Z be a matrix of the form $\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ when m is an integer not a square. If Z is expressed in the form $XY-YX$, where X, Y are rational matrices,*[3] *then the characteristic roots of X lie in the field $Q(\sqrt{M})$ where M is the norm of an element in $Q(\sqrt{m})$.*

It can further be shown that M can be chosen as an arbitrary norm in $Q(\sqrt{m})$. Combining this fact with Theorem 1 leads to the following result:

THEOREM 3. *Every negative norm in a quadratic field can be represented as* $\det(AB-BA)$.

EXAMPLES.

1. $A=\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 1 \\ n & 0 \end{pmatrix}$, $AB-BA=\begin{pmatrix} n-m & 0 \\ 0 & m-n \end{pmatrix}$.
2. $A=\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ n & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$,

$$AB - BA = \begin{pmatrix} n - m + mn & -2n \\ 2mn & -n + m - mn \end{pmatrix},$$

$$\det(AB - BA) = -[(n - m + mn)^2 - 4mn^2]$$
$$= -[(m - n + mn)^2 - 4m^2n].$$

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[3] This is always possible by a theorem of Shoda.

3. A random choice.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix}.$$

A has characteristic polynomial x^2-5x-2 and roots in $Q(\sqrt{(33)})$. B has characteristic polynomial x^2-6x+2 and roots in $Q(\sqrt{7})$. The commutator $AB-BA$ has determinant

$$-58 = -\text{norm}(31 + \sqrt{(33)})/4 = -\text{norm}(11 + 3\sqrt{7}).$$

REMARK. Zassenhaus observed that for matrices A with $A^2 = \det A \cdot I$ the relation $AL+LA=0$ holds. This can be generalized to the fact that the operator defined via A on the space of 2×2 matrices has the characteristic vector L with respect to the characteristic root trace A.

REFERENCE

1. O. Taussky, *A result concerning classes of matrices*, J. Number Theory **6** (1974), 64–71.

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