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IRREDUCIBLE REPRESENTATIONS OF LIE ALGEBRA EXTENSIONS

BY RICHARD E. BLOCK[1]

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This note announces three density theorems involving representations of Lie algebras and associative algebras. The first theorem describes the irreducible (possibly infinite dimensional) representations ρ of a Lie algebra $\mathfrak{g}$ with an ideal $\mathfrak{k}$ such that the restriction of ρ to $\mathfrak{k}$ has some absolutely irreducible quotient representation. The second result is an embedding theorem for the irreducible representations of the Weyl algebras $A_{n,C}$ over C ($A_{n,C} \cong C[t_1, \cdots, t_n, \partial/\partial t_1, \cdots, \partial/\partial t_n]$, the associative algebra of partial differential operators on n variables with coefficients in the polynomial ring $C[t_1, \cdots, t_n]$). Our result is a sort of algebraic analogue of the uniqueness of the Heisenberg commutation relations, and has an application to irreducible representations of nilpotent Lie algebras via Dixmier's theory [**5**]. The third theorem describes the differentiably simple algebras having a maximal ideal. This result unifies the author's theorem [**3**] on differentiably simple rings with a minimal ideal, and Guillemin's theorem [**7**], [**2**] on the structure of a nonabelian minimal closed ideal of a linearly compact Lie algebra.

1. In what follows, all algebras, tensor products etc., will be over an arbitrary given field Φ, unless otherwise stated. If the characteristic is prime, the Lie algebras considered will always be assumed restricted (=Lie p-algebra), and the same for their homomorphisms, ideals, etc. Also U will denote the universal enveloping algebra functor at characteristic 0, and the restricted universal enveloping algebra functor at prime characteristic. We shall take $\mathfrak{g}$ to be a given Lie algebra, and $\mathfrak{k}$ an ideal of $\mathfrak{g}$.

Recall that if V is a $\mathfrak{k}$-module with corresponding representation σ, then the stabilizer $\mathrm{St}(V, \mathfrak{g})$ of V in $\mathfrak{g}$ is defined [**1**], [**6**] by

$$\mathrm{St}(V, \mathfrak{g}) = \{x \in \mathfrak{g} \mid \exists \eta \in \mathrm{Hom}(V, V) \ni \sigma[x, y] = [\eta, \sigma y] \forall y \in \mathfrak{k}\}.$$

This is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$, and gives the analogue of the concept of stabilizer for group representations.

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At characteristic 0, Blattner [1] has shown that if V is a $\mathfrak{k}$-module which is absolutely irreducible (i.e. V is irreducible and has centralizer Φ), if $\mathfrak{h}=\mathrm{St}(V, \mathfrak{g})$, and if W is an irreducible $\mathfrak{h}$-module which as $\mathfrak{k}$-module is a direct sum of copies of V, then the induced $\mathfrak{g}$ module $U\mathfrak{g} \otimes_{U\mathfrak{h}} W$ is irreducible; Dixmier [6] has used this result to show that every irreducible $\mathfrak{g}$-module containing an absolutely irreducible $\mathfrak{k}$-submodule is isomorphic to such an induced module. It can be shown that the Blattner-Dixmier theorem remains valid in the (restricted) prime characteristic case.

We now turn to the more complicated situation of coinduced modules and irreducible $\mathfrak{g}$-modules with a *maximal* $\mathfrak{k}$-submodule. In topological considerations $\mathfrak{g}$ will have the discrete topology unless otherwise stated. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and W is an $\mathfrak{h}$-module then, regarding $U\mathfrak{g}$ as a $U\mathfrak{h}$, $U\mathfrak{g}$-bimodule, we get the coinduced $U\mathfrak{g}$-module $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$. (This gives the right adjoint to the forgetful functor from $U\mathfrak{g}$-modules to $U\mathfrak{h}$-modules; the left adjoint is given by the induced $U\mathfrak{g}$-module $U\mathfrak{g} \otimes_{U\mathfrak{h}} W$.) If W is a topological $U\mathfrak{h}$-module we give $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$ the finite-open topology.

LEMMA 1. *Let V be an absolutely irreducible $\mathfrak{k}$-module, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ containing $\mathrm{St}(V, \mathfrak{g})$, and W a topological $\mathfrak{h}$-module with a family $\{\pi_i\}_{i\in I}$ of $\mathfrak{k}$-maps $\pi_i: W \to V$ such that the topology on W is that induced by $\{\pi_i\}_{i\in I}$ (i.e. the weakest topology making all π_i continuous) where V is discrete. Then the coinduced $U\mathfrak{g}$-module, $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$, is topologically irreducible if (and only if) W is.*

Blattner [1], [2] has proved a related result for the case in which V is linearly compact and topologically absolutely irreducible and W (as $\mathfrak{k}$-module) is a product of copies of V.

Note that $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$ above has a maximal $\mathfrak{k}$-submodule. The following seems to be the first result in the converse direction, i.e. describing the irreducible $\mathfrak{g}$-modules having a maximal $\mathfrak{k}$-submodule.

THEOREM 1. *Let M be an irreducible $\mathfrak{g}$-module having a (maximal) $\mathfrak{k}$-submodule N such that the quotient $V=M/N$ is absolutely irreducible (as a $\mathfrak{k}$-module); give V the discrete topology, and let $\mathfrak{h}=\mathrm{St}(V, \mathfrak{g})$. Then there is a topological $\mathfrak{h}$-module W which as a $\mathfrak{k}$-module is a dense topological submodule of a product of copies of V such that M is isomorphic to a dense submodule of the coinduced module $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$.*

Here W can be taken to be the quotient of M modulo the largest $\mathfrak{h}$-submodule contained in N. It follows from the theorem that the annihilator (in $U\mathfrak{g}$) of M equals the largest (two-sided) ideal of $U\mathfrak{g}$ contained in $P(U\mathfrak{g})$ where P is the annihilator (in $U\mathfrak{h}$) of W. By Quillen's lemma [8], the hypothesis that the irreducible module V be absolutely irreducible can be deleted if $\mathfrak{k}$ is finite dimensional and Φ is algebraically closed.

For brevity, the results above have been stated in less than their full generality. We remark in particular that they have a very useful extension to analogous results where one is given an associative algebra B (with 1) in place of $\mathfrak{k}$, and an action of $\mathfrak{g}$ on B by derivations (i.e. a homomorphism ζ of $\mathfrak{g}$ to Der B). Then one can form the smash (or semidirect) product $B\#U\mathfrak{g}$ (see [**9**]). Analogues of Lemma 1 and Theorem 1 (with V a B-module, W a $B\#U\mathfrak{h}$-module, and M a $B\#U\mathfrak{g}$-module) can be shown for the coinduced $B\#U\mathfrak{g}$-module $\mathrm{Hom}_{B\#U\mathfrak{h}}(B\#U\mathfrak{g}, W)$ which in fact is $U\mathfrak{g}$-module isomorphic (and homeomorphic) to $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$. Similarly we can extend the Blattner-Dixmier theorem to a result on the induced $B\#U\mathfrak{g}$-module $B\#U\mathfrak{g} \otimes_{B\#U\mathfrak{h}} W$ (which is $U\mathfrak{g}$-module isomorphic to $U\mathfrak{g} \otimes_{U\mathfrak{h}} W$). These results on $B\#U\mathfrak{g}$-modules can be applied in the study of differentiably irreducible modules [**4**] and are used in the proofs of Theorems 2 and 3 below.

2. The Weyl algebra A_n $(n \geqq 0)$ over Φ is the associative algebra with unit with generators $x_1, \cdots, x_n, y_1, \cdots, y_n$ subject to the relations

$$[x_i, x_j] = [y_i, y_j] = 0, \qquad [x_i, y_j] = \delta_{ij} \qquad (i, j = 1, \cdots, n).$$

A faithful representation ρ of A_n in $\Phi[[t_1, \cdots, t_n]]$ (formal power series in n indeterminates) is given via $\rho(x_i) = \partial/\partial t_i$, $\rho(y_i) = \mu(t_i)$, where $\mu(t_i)$ denotes the multiplication by t_i. Then $\Phi[t_1, \cdots, t_n]$ is an irreducible A_n-submodule.

In order to state our result on the irreducible representations of A_n, we define a *special automorphism* θ of A_n to be an automorphism such that for $i = 1, \cdots, n$ and some scalars $\alpha_1, \cdots, \alpha_n$, either $\theta x_i = x_i$ and $\theta y_i = y_i - \alpha_i 1$ or $\theta x_i = -y_i$ and $\theta y_i = x_i - \alpha_i 1$. Given such a θ, $\rho\theta$ is a representation of A_n in $\Phi[[t_1, \cdots, t_n]]$, i.e. we get an A_n-module, $\Phi[[t_1, \cdots, t_n]]_\theta$, where for each i either x_i acts by $\partial/\partial t_i$ and y_i by $\mu(t_i) - \alpha_i 1$, or y_i acts by $-\partial/\partial t_i$ and x_i by $\mu(t_i) - \alpha_i 1$.

THEOREM 2. *Let Φ be an algebraically closed nondenumerable field of characteristic* 0, *A_n the Weyl algebra over Φ (with generators $x_1, \cdots, x_n$, $y_1, \cdots, y_n$ as above), and M an irreducible A_n-module. Then M is isomorphic to a submodule of $\Phi[[t_1, \cdots, t_n]]_\theta$ for some special automorphism θ.*

Every nonzero submodule of $\Phi[[t_1, \cdots, t_n]]_\theta$ is dense in $\Phi[[t_1, \cdots, t_n]]$. The polynomials form an irreducible submodule of $\Phi[[t_1, \cdots, t_n]]_\theta$, but an abundance of others ($|\Phi|$ nonisomorphic ones) can be exhibited.

Combining Theorem 2 with a theorem of Dixmier [**5**] (whose work on the irreducible representations of nilpotent finite-dimensional Lie algebras in a sense reduces their classification to that of the irreducible representations of A_n), we get the following.

COROLLARY 1. *With Φ as in Theorem 2, let τ be an irreducible representation of a finite-dimensional nilpotent Lie algebra $\mathfrak{g}$. Then for some $n \geqq 0$, τ is equivalent to a subrepresentation of a representation ψ of $\mathfrak{g}$ acting in $\Phi[[t_1, \cdots, t_n]]$ such that the sets $\{\psi(x) | x \in \mathfrak{g}\}$ and $\{\partial/\partial t_i, \mu(t_i) | i=1, \cdots, n\}$ generate the same subalgebra of endomorphisms of $\Phi[[t_1, \cdots, t_n]]$.*

3. The next theorem determines the structure of certain algebras with no proper ideal invariant under a given family of derivations. Before stating it we need some preliminary notions. Suppose S is a (not necessarily associative) simple algebra. The centroid (or multiplication centralizer) Γ of S is the subalgebra of elements of $\mathrm{Hom}_\Phi(S, S)$ which commute with all the left and right multiplications of S. Since S is simple, Γ is a field, and S is also an algebra over Γ. The scalar extension of the given Lie algebra $\mathfrak{g}$ to Γ will be denoted by $\mathfrak{g}_\Gamma$. Since $U\mathfrak{g}_\Gamma$ is a coalgebra, $\mathrm{Hom}_\Gamma(U\mathfrak{g}_\Gamma, S)$ is an algebra under the convolution multiplication, and $\mathrm{Hom}_\Gamma(U\mathfrak{g}_\Gamma, S)$ satisfies any multilinear identities that S does. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}_\Gamma$ and $\mathfrak{h}$ acts on S (as an algebra over Γ) by derivations, then it may easily be shown that $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}_\Gamma, S)$ is a subalgebra of $\mathrm{Hom}_\Gamma(U\mathfrak{g}_\Gamma, S)$. By a topological $\mathfrak{g}$-algebra we shall mean a topological algebra on which $\mathfrak{g}$ acts by continuous derivations. If S is a topological algebra then $\mathrm{Hom}_\Gamma(U\mathfrak{g}_\Gamma, S)$ is a topological $\mathfrak{g}$-algebra. A topological $\mathfrak{g}$-algebra will be called topologically $\mathfrak{g}$-simple if it has no proper closed ideal invariant under $\mathfrak{g}$.

THEOREM 3. *Let R be a topological $\mathfrak{g}$-algebra (possibly nonassociative or discrete) which is topologically $\mathfrak{g}$-simple. Suppose R has a closed maximal ideal N, write Γ for the centroid of the (simple) algebra $S=R/N$, and assume that each γ in Γ is continuous (on S) and that Γ is separable algebraic over Φ. Then there is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\Gamma$ and a continuous isomorphism (of $\mathfrak{g}$-algebras) of R onto a dense $\mathfrak{g}$-subalgebra of* $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}_\Gamma, S)$.

It can easily be seen that $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}_\Gamma, S)$ is isomorphic as a topological algebra to $S[[X_j]]_{j\in J}$, the formal power series in the indeterminates $\{X_j\}_{j\in J}$ with coefficients in S, where the indeterminates are p-truncated ($X_j^p=0$ for all j) in case the characteristic is a prime p. Here the cardinality of J equals the codimension of $\mathfrak{h}$ in $\mathfrak{g}_\Gamma$.

We mention two important special cases as applications of Theorem 3. First, suppose that R is a discrete (not necessarily associative) algebra. Recall that R is called differentiably simple if $R^2 \neq 0$ and if there is a set D of derivations of R (one might as well take $D=\mathrm{Der}\, R$) such that R has no proper D-ideal, i.e. ideal invariant under D. For example, a nonabelian minimal ideal of a Lie algebra is differentiably simple. The main result of **[3]** says that if R is differentiably simple and has a minimal (two-sided) ideal then R has a (unique) maximal ideal N and either R is simple (i.e. $R=R/N$) or Φ has prime characteristic p and R is isomorphic to the algebra of p-truncated polynomials in some finite set of indeterminates with coefficients in the simple algebra R/N. The special case when R is finite dimensional and Φ is perfect is thus generalized by Theorem 3 (with $\mathfrak{g}=\operatorname{Der} R$).

The second application concerns topological Lie algebras which are linearly compact. Guillemin [7] has proved that if R is a nonabelian minimal closed ideal of such a Lie algebra then R has a (unique) closed maximal ideal N, the centroid Γ of the simple Lie algebra R/N is a finite-dimensional extension of Φ, and if, in addition, Φ has characteristic 0, then R is isomorphic to the algebra $(R/N)[[X_j]]_{j\in J}$ of formal power series for some finite set J. In our case we can obtain Guillemin's theorem (and a little more) as a corollary of Theorem 3; moreover the result remains valid at prime characteristic (with the X_j p-truncated).

Blattner [2] has given another proof of Guillemin's theorem when Φ is algebraically closed (actually when $\Gamma=\Phi$) at characteristic 0. Our proof to an extent resembles Blattner's.

REFERENCES

1. R. J. Blattner, *Induced and produced representations of Lie algebras*, Trans. Amer. Math. Soc. **144** (1969), 457–474. MR **46** #7338a.

2. ———, *A theorem of Cartan and Guillemin*, J. Differential Geometry **5** (1970), 295–305. MR **46** #7338b.

3. R. E. Block, *Determination of the differentiably simple rings with a minimal ideal*, Ann. of Math. (2) **90** (1969), 433–459. MR **40** #4319.

4. ———, *Modules over differential polynomial rings*, Bull. Amer. Math. Soc. **79** (1973), 729–733.

5. J. Dixmier, *Représentations irréductibles des algèbres de Lie nilpotentes*, An. Acad. Brasil. Ci. **35** (1963), 491–519. MR **32** #165.

6. ———, *Sur les représentations induites des algèbres de Lie*, J. Math. Pures Appl. **50** (1971), 1–24.

7. V. Guillemin, *A Jordan-Hölder decomposition for a certain class of infinite dimensional Lie algebras*, J. Differential Geometry **2** (1968), 313–345. MR **41** #8481.

8. D. Quillen, *On the endomorphism ring of a simple module over an enveloping algebra*, Proc. Amer. Math. Soc. **21** (1969), 171–172. MR **39** #252.

9. M. E. Sweedler, *Hopf algebras*, Mathematics Lecture Notes Series, Benjamin, New York, 1969. MR **40** #5705.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92502