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SINGULARITIES AND BORDISM OF q-PLANE FIELDS AND OF FOLIATIONS[1]

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1. **Introduction.** Let $\mathfrak{P}\mathfrak{N}_n(q)$ (resp. $\mathfrak{P}\mathfrak{N}_n^{or}(q)$) be the bordism group of n-dimensional smooth manifolds with arbitrary (resp. oriented) q-plane fields, and let $\mathfrak{P}\Omega_n(q)$ and $\mathfrak{P}\Omega_n^{or}(q)$ denote the corresponding groups based on oriented manifolds. In this paper we present a method which allows us in many cases to determine these groups. We use the forgetful homomorphism $f_{\mathfrak{P}}: \mathfrak{P}\mathfrak{N}_n(q) \to \mathfrak{N}_n(BO(q))$ (resp. $f_{\mathfrak{P}}: \mathfrak{P}\mathfrak{N}_n^{or}(q) \to \mathfrak{N}_n(BSO(q))$, resp. $f_{\mathfrak{P}}: \mathfrak{P}\Omega_n^{(or)}(q) \to \Omega_n(B(S)O(q))$), which assigns to the bordism class of a q-plane field the bordism class of (a classifying map of) the underlying vector bundle. Our point of departure is the following observation. If ξ is a q-dimensional vector bundle over an n-manifold M and $n \geqq 2q-3$, then it is always possible to find a vector bundle homomorphism $h: \xi \to TM$ which is injective outside of a $(q-1)$-dimensional submanifold S of M, and such that the kernel of h is 1-dimensional at every point of S. We investigate the behavior of h at such a singularity and obtain criteria as to when it is possible to cancel S without getting out of the original bordism class.

If M is closed and ξ is isomorphic to a q-dimensional subbundle of TM, then the element $TM-\xi$ in the K-theory of M can be represented by an $(n-q)$-dimensional bundle, and hence the class $[M, \xi]$ in the bordism of $B(S)O$ satisfies the following *vanishing condition*:

(V) all Whitney numbers of $[M, \xi]$ containing some $w_i(TM-\xi)$, $i > n-q$, as a factor, vanish.

Conversely we obtain

THEOREM 1. *Let $n > 2q-2$. Then under all four orientedness assumptions $[M, \xi]$ lies in the image of $f_{\mathfrak{P}}$ if and only if condition* (V) *is satisfied. Furthermore, the kernel as well as the cokernel of $f_{\mathfrak{P}}$ are finite groups consisting entirely of elements of order* 2.

A stable version of the first statement for the case of $\mathfrak{N}_n(BO(q))$ has previously been obtained by R. Stong [11] by other methods.

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COROLLARY 1. *$\mathfrak{P}\mathfrak{N}_n(q)$ and $\mathfrak{P}\mathfrak{N}_n^{or}(q)$ are finite vector spaces over Z_2. $\mathfrak{P}\Omega_n(q)$ and $\mathfrak{P}\Omega_n^{or}(q)$ are finitely generated groups whose torsion consists entirely of elements of order 2 or possibly 4.*

These results can be sharpened in many cases to give a complete description of our groups. For example

THEOREM 2. *$f_{\mathfrak{P}}$ gives an isomorphism between $\mathfrak{P}\mathfrak{N}_n(q)$ and the subgroup of $\mathfrak{N}_n(BO(q))$ consisting of all elements $[M, \xi]$ which satisfy condition* (V) *above.*

For a determination of the plane field bordism groups with other orientedness assumptions see also [6] for $q=1$ and [7] for $q=2$.

If we also take vanishing conditions for the Pontrjagin numbers into account we may in many cases avoid the restriction $n>2q-2$. This can be done either by also considering singularities with higher dimensional kernel, or by applying our approach to complementary $(n-q)$-plane fields. Thus, e.g., Corollary 1 and Theorem 2 turn out to hold whenever $0\leqq q\leqq n$, the latter as a consequence of the following duality result.

THEOREM 3. *If $0\leqq q\leqq n$, there is a natural isomorphism $\mathfrak{P}\mathfrak{N}_n(q)\cong \mathfrak{P}\mathfrak{N}_n(n-q)$ obtained by taking complements.*

This is not a priori obvious since the standard bordism relation for q-plane fields induces a different (stabilized) bordism relation for the complementary $(n-q)$-plane fields.

Next define $\mathfrak{F}\mathfrak{N}_n(q)$, $\mathfrak{F}\mathfrak{N}_n^{or}(q)$, $\mathfrak{F}\Omega_n(q)$ and $\mathfrak{F}\Omega_n^{or}(q)$ to be the bordism groups of closed n-manifolds with smooth q-codimensional foliations, satisfying the indicated (co)-orientedness conditions. For $q\geqq 2$ Thurston [13] has shown recently that a foliation on a compact manifold M is essentially given by an $(S)\Gamma$-structure γ on M (in the sense of Haefliger [3]) together with a bundle monomorphism from the normal bundle $\nu(\gamma)$ of γ into TM.[2] Thus when we compare our foliation bordism groups with the corresponding usual bordism groups of Haefliger's classifying spaces $B\Gamma(q)$ and $BS\Gamma(q)$, we are only confronted with a plane field problem and can apply our approach. We obtain for the forgetful homomorphism $f_{\mathfrak{F}}: \mathfrak{F}\mathfrak{N}_n^{(or)}(q)\to\mathfrak{N}_n(B(S)\Gamma(q))$, resp. $f_{\mathfrak{F}}: \mathfrak{F}\Omega_n^{(or)}(q)\to\Omega_n(B(S)\Gamma(q))$:

THEOREM 1′. *Let $q\geqq 2$ and $n>2q-2$. Then under all four orientedness assumptions, an element $[M,\gamma]$ of the n-dimensional bordism group of $B(S)\Gamma(q)$ lies in the image of $f_{\mathfrak{F}}$ if and only if the vanishing condition* (V) *is satisfied by the normal bundle $\xi=\nu(\gamma)$. Furthermore the kernel as well as the cokernel of $f_{\mathfrak{F}}$ are finite groups consisting entirely of elements of order 2.*

[2] ADDED IN PROOF. More recent work of Thurston implies that the results of this paper still hold for foliations of codimension $q=1$.

This contrasts with the fact that the foliation bordism groups themselves need not even be countably generated. E.g., $\mathscr{F}\Omega_{2q+1}^{or}(q)$ surjects onto $\boldsymbol{R}$ for even positive q (see [**14**]).

THEOREM 2′. *If* $q\geqq 2$ *and* $n\geqq 2q-2$, *then* $f_{\mathfrak{F}}$ *gives an isomorphism between* $\mathfrak{F}\mathfrak{N}_n(q)$ *and the subgroup of* $\mathfrak{N}_n(B\Gamma(q))$ *consisting of all elements* $[M, \gamma]$ *for which the normal bundle* $\xi=\nu(\gamma)$ *satisfies condition* (V).

As a corollary to the proof we have

THEOREM 4. *For* $q\geqq 1$, $n\geqq 2q-2$, *every q-plane field on a closed n-manifold is bordant (in* $\mathfrak{P}\mathfrak{N}_n(q)$*) to one which is transversal to a foliation of codimension q.*

The case $q=1$ (where Thurston's results are not available[3]) was settled in [**8**] by an explicit construction of enough foliations to generate $\mathfrak{P}\mathfrak{N}_n(1)$ by their normal linefields.

Finally, note that the singularity approach can also be fruitfully applied to the bordism of manifolds with tangent q-frames, or to the bordism of immersions and, more generally, of k-mersions. More details on this point will appear elsewhere (see also [**9**]).

I would like to thank Peter Landweber for many helpful references.

2. **The singularity isomorphism.** Let $\mathfrak{N}_n(BO(q), \mathfrak{P})$ (resp. $\mathfrak{N}_n(B\Gamma(q), \mathfrak{F})$) be the bordism group of triples (M, ξ, h') (resp. (M, γ, h')) where M is a compact smooth n-manifold, ξ is a q-plane bundle over M (resp. γ is a $\Gamma(q)$-structure on M, and we write ξ for its normal bundle $\nu(\gamma)$), and $h'\colon \xi|\partial M\to T(\partial M)$ is a bundle monomorphism. Denote the normal bundle map from $\mathfrak{N}_n(B\Gamma(q), \mathfrak{F})$ into $\mathfrak{N}_n(BO(q), \mathfrak{P})$ by ν_*.

Now for $0\leqq p\leqq q$ consider the $p\cdot(n-q+p)$-codimensional submanifold A_p of the total space of the homomorphism bundle $\mathbf{Hom}(\xi, TM)$ where $A_p=\bigcup_{x\in M} A_p(x)$ and $A_p(x)=\{g\colon \xi_x\to T_xM \mid g \text{ linear, } \dim(\ker g)=p\}$ (cf. [**5**, p. 120]). If $n\geqq 2q-3$, or equivalently, if $2(n-q+2)>n$, then, by transversality we can extend h' to a vector bundle morphism $h\colon \xi\to TM$ which, as a section in $\mathbf{Hom}(\xi, TM)$, goes entirely into $A_0\cup A_1$ and intersects A_1 transversally. Denote by S the closed $(q-1)$-dimensional submanifold $h^{-1}(A_1)$ of the interior of M. Since $h|S$ has constant rank, there are canonical vector bundles **Ker**, **Coker**, and **Im** over S of dimension 1, $n-q+1$, and $q-1$, respectively, where e.g., the fiber of **Ker** at $x\in S$ is the kernel of $h_x\colon \xi_x\to T_xM$. These bundles are related to $\xi|S$, $TM|S$ and the

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[3] See footnote 2.

normal bundle $\nu(S, M)$ of S in M by the following isomorphisms (which are canonical up to homotopy)

$$(1)\qquad \begin{aligned} \xi \mid S &\cong \mathbf{Im} \oplus \mathbf{Ker}, \\ TM \mid S &\cong \mathbf{Im} \oplus \mathbf{Coker}, \\ \nu(S, M) &\cong \mathbf{Hom}(\mathbf{Ker}, \mathbf{Coker}); \end{aligned}$$

and consequently

$$(2)\qquad \mathbf{i}\colon \mathbf{Im} \oplus \mathbf{Coker} \cong TS \oplus \mathbf{Hom}(\mathbf{Ker}, \mathbf{Coker}).$$

Associating the bordism class of $(S, \mathbf{Ker}, \mathbf{Coker})$ to the class of (M, ξ, h'), we obtain a well-defined homomorphism

$$\begin{aligned} \sigma\colon \mathfrak{N}_n(BO(q), \mathfrak{P}) &\to \mathfrak{N}_{q-1}(BO(1) \times BO(n-q+1)) \\ &\cong \mathfrak{N}_{q-1}(BO(1) \times BO(q)), \end{aligned}$$

provided $n \geqq 2q-2$. Similarly σ is defined on the relative bordism groups $\mathfrak{N}_n(BSO(q), \mathfrak{P})$ and $\Omega_n(B(S)O(q), \mathfrak{P})$ corresponding to the other orientation cases.

We will say that an element $x=[S, \zeta, \eta]$ of $\mathfrak{N}_{q-1}(BO(1)\times BO(q))$ satisfies *condition O_b (resp. O_m) for* (n, q) if all those Whitney numbers vanish which either involve $w_1(S)+(n-q)w_1(\zeta)$ as a factor or which are made up entirely by a positive number of factors of the form $n \cdot w_{2k}(S)^2$ or $n \cdot w_{2k}(\eta)^2$, $k \geqq 0$ (resp. if all Whitney numbers of x involving $w_1(S)+(n-q+1)w_1(\zeta)+w_1(\eta)$ vanish).

THEOREM 5. *Let $n>2q-2$. Then under all four orientedness assumptions σ is an isomorphism into $\mathfrak{N}_{q-1}(BO(1)\times BO(q))$. An element x of*

$$\mathfrak{N}_{q-1}(BO(1) \times BO(q))$$

lies in the image of $\mathfrak{N}_n(BO(q), \mathfrak{P})$ (resp. $\mathfrak{N}_n(BSO(q), \mathfrak{P})$, resp. $\Omega_n(BO(q), \mathfrak{P})$, resp. $\Omega_n(BSO(q), \mathfrak{P})$) under σ if and only if x is arbitrary (resp. x satisfies condition O_b, resp. O_m, resp. O_b and O_m, for (n, q)).

If in addition $q \geqq 2$, then in all four orientedness cases $\sigma \circ \nu_*$ is also an isomorphism onto the image of σ.

In particular, for fixed q the relative bordism groups of a given orientation type depend only on the parity of n.

In the proof we use generalized surgery with core manifolds of dimension q or 1 or 2. The construction extends to the case of Γ-structures since the normal bundle map $\nu\colon B\Gamma(q)\to BO(q)$ has a q-connected homotopic fiber [3].

The relevance of Theorem 5 stems from the following commutative diagram and its analogues in the other orientation cases

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathfrak{F}\mathfrak{N}_n(q) & \xrightarrow{f_{\mathfrak{F}}} & \mathfrak{N}_n(B\Gamma(q)) & \xrightarrow{j} & \mathfrak{N}_n(B\Gamma(q), \mathfrak{F}) & \xrightarrow{\partial} & \mathfrak{F}\mathfrak{N}_{n-1}(q) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \nu_*} & \\
\cdots \longrightarrow & \mathfrak{P}\mathfrak{N}_n(q) & \xrightarrow{f_{\mathfrak{P}}} & \mathfrak{N}_n(BO(q)) & \xrightarrow{j} & \mathfrak{N}_n(BO(q), \mathfrak{P}) & \xrightarrow{\partial} & \mathfrak{P}\mathfrak{N}_{n-1}(q) & \longrightarrow \cdots \\
(3) & & & & & \downarrow{\scriptstyle \sigma} & & & \\
& & & & & \mathfrak{N}_{q-1}(BO(1) \times BO(q)) & & &
\end{array}$$

Here the forgetful homomorphisms j and ∂ make the horizontal sequences exact.

In order to describe $\sigma \circ j$ in terms of Whitney numbers, assume M to be closed in the discussion above. In a Whitney number of (S, **Ker**, **Coker**) eliminate first $w(S)$, and then $w(\mathbf{Coker})$, using (1) and (2), and apply the identity

$$w_1(\mathbf{Ker})^k \cdot (w(TM - \xi)^\alpha w(M)^\beta \mid S)[S] = w_{n-q+1+k}(TM - \xi) w(TM - \xi)^\alpha w(TM)^\beta [M],$$

where α, β are multi-indices.

Now Theorem 5 implies Theorem 1, Corollary 1 and Theorem 1′. To obtain a full description of the bordism groups of q-plane fields, it remains only to determine the image of j, or equivalently, of $\sigma \circ j$ (and to check for possible 4-torsion in $\mathfrak{P}\Omega_n^{(or)}(q)$). For example, a geometric construction yields

THEOREM 6. *For $n \geqq 2q-2$, the homomorphism $\sigma \circ j: \mathfrak{N}_n(BO(q)) \to \mathfrak{N}_{q-1}(BO(1) \times BO(q))$ is onto.*

Thus, if no orientation conditions are imposed, the lower horizontal line in diagram (3) breaks down into short exact sequences ($\partial = 0$), and so does the upper line since the middle homomorphism ν_* is surjective here (compare [**1**]). This proves Theorems 2 and 2′. Theorem 4, or equivalently, the surjectivity of the left hand homomorphism ν_*, follows immediately.

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