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COMPLETE CONVEX HYPERSURFACES OF A HILBERT SPACE

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A *complete convex hypersurface* of a (separable) Hilbert space H is a codimension one C^∞ submanifold of H, which is complete as a metric subspace of H and such that $M=\partial K$, where K is a (closed) convex set with nonvoid interior. For each $p \in M$ let $\nu(p)$ be the unit normal vector which points to the interior of K. In this way we define the *Gauss map* $\nu: M \to \Sigma$ from M into the unit sphere Σ of H. This is a C^∞ map and its derivative at each point $p \in M$ is selfadjoint. We say that *M bounds a half-line* if there exists a half-line $\{p+tv; t \geqq 0\}$ contained in the interior of K. In the finite dimensional case the condition that M bounds a half-line is equivalent to that M is unbounded. In the infinite dimensional case this is not true, as the following simple example shows. Let A be a compact positive definite selfadjoint operator in H and set $M=\{x \in H; \langle A(x), x\rangle = 1\}$. It is not difficult to prove that M is an unbounded positively-curved convex hypersurface and that M does not bound any half-line.

In this note we announce some properties of a complete convex hypersurface M of a Hilbert space. Theorem A characterizes the three possible boundedness situations (bounded, unbounded and bounding a half-line, unbounded and bounding no half-line) in terms of the Gauss map of M. Theorem B gives a necessary and sufficient condition for M to be a pseudograph (see definition below) over one of its tangent hyperplanes. Theorem C is the analogue of the Bonnet-Myers theorem for hypersurface of a Hilbert space. These results are part of my doctoral dissertation. I wish to thank my advisor Professor Manfredo do Carmo for suggesting these problems and for helpful conversations.

THEOREM A. *Let M be a complete convex hypersurface of a Hilbert space H. Then*:

(1) *M is bounded iff the Gauss map $\nu: M \to \Sigma$ is onto.*

(2) *M is unbounded and bounds a half-line iff the image of the Gauss map is contained in a hemisphere.*

(3) *M is unbounded and does not bound any half-line iff the image of the Gauss map is dense and has void interior.*

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Before stating Theorem B, we need to rephrase slightly the definition of pseudograph given in [**2**]. A convex hypersurface M is a *pseudograph* over the tangent hyperplane F when:

(a) M lies in one of the closed half-spaces determined by F,

(b) let $\pi: M \to F$ be the orthogonal projection and set $A = \pi(M)$. Then over the interior int A, M is the graph of a C^∞ function,

(c) for every $a \in A - \text{int } A$, $M \cap \pi^{-1}(a)$ is a closed half-line,

(d) for every hyperplane L above F, $M \cap L$ is bounded.

In the case that M is finite dimensional, the above reduces to the definition given in [**2**].

THEOREM B. *Let M be a complete convex hypersurface of a Hilbert space H. Then M is unbounded and* $\text{int}(\nu(M)) \neq \varnothing$ *iff M is a pseudograph over one of its tangent hyperplanes $TM_p \neq M$.*

THEOREM C (THE BONNET-MYERS THEOREM FOR HILBERT HYPERSURFACE). *Let M be a complete connected hypersurface of a Hilbert space H. If the sectional curvatures of M are all bounded away from zero (i.e. there exists a $\delta > 0$ such that for every $p \in M$ and every two-plane section $\sigma \subset TM_p$ one has $K(\sigma) \geqq \delta$) then M is bounded, the diameter ρ of M satisfies $\rho \leqq \pi/\sqrt{\delta}$ and the Gauss map is a diffeomorphism.*

REMARK 1. Theorem B should be compared with a theorem of H. H. Wu [**2**]. It should be remarked that Wu also proved that if M is a complete convex hypersurface of R^n, then $\text{int}(\nu(M)) = \text{int}(\overline{\nu(M)})$.

Theorem A shows that in the infinite dimensional case, this equality does not hold and we may have the extremal case in which $\text{int}(\nu(M)) = \varnothing$ and $\text{int}(\overline{\nu(M)}) = \Sigma$. This explains why we need the condition $\text{int}(\nu(M)) \neq \varnothing$ in Theorem B, in contrast with Wu's theorem, where no such condition is required.

REMARK 2. The hypothesis of Theorem B is implied by the following condition on the sectional curvature of M (see [**1**]): The sectional curvatures of M are everywhere nonnegative and at some point $p \in M$ are all bounded away from zero. Thus in the finite dimensional case, Theorem B reduces to Wu's theorem.

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