BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, May 1974

LOCALLY PRIME ARCS WITH FINITE PENETRATION INDEX

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Communicated by O. G. Harrold, September 29, 1973

Introduction. Let k be an oriented arc in R^3, which has an isolated wild point p at which the penetration index P of k is finite and no less than three. We say that k is locally prime at p if there is a tame closed 3-cell neighbourhood U of p which meets k on its boundary in exactly P points, such that if Γ is any cube in Int U which meets k on its boundary in two points, then it is impossible for k to be knotted in Γ; that is, $\pi_1(\Gamma-k)$ must be free cyclic. For example, the arc shown in Figure 2 of [2] is not locally prime at its wild endpoint, whereas the arcs $A_1, A_2, \cdots$ of [1] are all locally prime at their respective endpoints.

The purpose of this paper is to announce the existence of a factorisation theorem for arcs which are not locally prime at their isolated wild points, to the effect that each nonlocally-prime arc can be decomposed into the "product" of a locally prime arc with a sequence of tame knots, and that this decomposition is unique; this extends the 1961 result of Fox and Harrold [5]. The proofs, which rely heavily on (sometimes tortuous) cutting and pasting arguments, will appear in another paper.

The second author is presently working on a more general factorisation theorem for arcs with isolated wild points and finite penetration index, to the effect that any such arc may be written as a finite "product" of arcs, so that each term in the product is an arc which is not itself the composite of two other wild arcs.

We have not yet explored the possibility of a more general theorem for arcs which are locally knotted in the sense of [9].

Results. Except in Theorem 1, we assume that our knots and arcs have only one wild point; the generalisation to knots and arcs with finitely many wild points is easily accomplished.

A *cube* is any tame set homeomorphic to I^3; it is *well-placed* (with respect to k) if it meets k on its boundary in exactly two points. If Γ is a well-placed cube, we say that *k represents the knot κ in Γ* if joining the endpoints of $k\cap\Gamma$ by a simple arc on Bd Γ yields an oriented knot equivalent to κ in R^3, and Γ is a *k-prime* cube if k represents a (nontrivial) prime knot in Γ. We use the following fact from [10]:

LEMMA A. *If k represents the knot $\pi_1\#\cdots\#\pi_m$ in Γ (where each π_i is an oriented tame prime knot, and $\#$ denotes tame knot multiplication),*

AMS (MOS) subject classifications (1970). Primary 55A30.

there exist m disjoint well-placed cubes in Int Γ, *say* $\Gamma_1, \cdots, \Gamma_m$, *such that k represents* π_i *in* Γ_i, *for each i.*

We use the word "3-cell" in a restricted sense, in that a cube is a *3-cell* only when it is a neighborhood of p and meets k on its boundary in exactly P points; we also assume that our 3-cells are "small enough", in that if E is a 3-cell and C a cube which is a neighborhood of p, then Bd C contains at least P points of k. An *admissible sequence* of 3-cells for k is a sequence $E_0 \supset E_1 \supset \cdots$ of 3-cells for which $E_{i+1} \subset \operatorname{Int} E_i$ and $\bigcap E_i = p$.

The following lemmas are crucial; in each, $E_0 \supset E_1 \supset \cdots$ is an admissible sequence for k, and our well-placed cubes are assumed to lie in Int E_0.

LEMMA B. *The fundamental group homomorphisms induced by the inclusions of* $\operatorname{Bd} E_i - k$ *in* $E_i - k - \operatorname{Int} E_{i+1}$, $E_0 - k - \operatorname{Int} E_i$ *and* $E_i - k$, *and of* $\operatorname{Bd} E_{i+1} - k$ *in* $E_i - k - \operatorname{Int} E_{i+1}$, *are monomorphisms*, $i \geqq 0$.

LEMMA C. *Let* Γ *be a k-prime cube. There exists a well-placed cube* Γ^* *in* $\operatorname{Int}(E_i - E_{i+1})$, *some i, in which k represents the same prime as it does in* Γ. *Moreover, we may assume that* Γ_1^* *and* Γ_2^* *are disjoint if* Γ_1 *and* Γ_2 *are disjoint k-prime cubes.*

LEMMA D. *We fix an index i and work in* $\operatorname{Int}(E_i - E_{i+1})$. *Let* $\Gamma_1, \cdots, \Gamma_n$ *be a family of disjoint well-placed cubes, and let* Γ *be a k-prime cube. There exists a well-placed cube* Γ^* *in which k represents the same prime as in* Γ, *and such that* Bd Γ^* *meets none of the surfaces* Bd $\Gamma_1, \cdots,$ Bd Γ_n. *Moreover, we may assume that* Γ_a^* *and* Γ_b^* *are disjoint if* Γ_a *and* Γ_b *are disjoint k-prime cubes.*

One consequence of Lemmas B and C is the following: If $\pi_1(E_i - E_{i+1} - k)$ is free for each i, then there are no k-prime cubes in Int E_0 (for the existence of a k-prime cube in Int E_0 implies that $\pi_1(E_i - E_{i+1} - k)$ has a nonfree knot group as a subgroup, for some i).

Another consequence of Lemmas B and C, and the finiteness theorem of [6, p. 48], is that there are only finitely many disjoint k-prime cubes in $\operatorname{Int}(E_i - E_{i+1})$, for any i. Note that this result is not true for tame arcs, but [5] shows it is true for Wilder arcs.

DEFINITION. Let κ be an infinite sequence of (nontrivial) oriented prime tame knots, and k_0 an arc which has finite penetration index at its single wild point p_0. The arc k_1 with wild point p_1 (with $P(k_1, p_1) = P(k_0, p_0) \geqq 3$) is (locally) the *product of* k_0 *with the sequence* κ, written $k_1 = k_0 \# \kappa$, if there exist infinitely many disjoint k_1-prime cubes Γ_i converging to p_1, such that

(i) the prime knot types represented by k_1 in this sequence of cubes are in one-one onto correspondence with the prime knots of the sequence κ, and

(ii) if γ_i is an oriented arc running from the starting point of $k_1 \cap \Gamma_i$

along Bd Γ_i to the endpoint of $k_1 \cap \Gamma_i$, then the consistently oriented arc $\bigcup (\gamma_i \cup (k_1 - \Gamma_i))$ has the same oriented local type at p_1 as k_0 has at p_0.

Figure 2 of **[2]** shows the (nonoriented) product of Example 1.2 of **[4]** with an infinite sequence of trefoil knots.

Note that if p_0 is an interior point of k_0, and κ^* the Wilder arc formed from the sequence κ, then $k_0 \# \kappa$ is the interior composite $k_0 \$_p \kappa^*$ (cf. **[7**, V.C. 7]).

Definition. The arc k is *locally prime at p* if it cannot be expressed as the product of another arc with an infinite sequence of nontrivial oriented prime knots.

An arc which is not locally prime at an interior wild point is locally knotted at that point **[9]**.

Theorem 1. *The following arcs are locally prime at their respective wild points*:

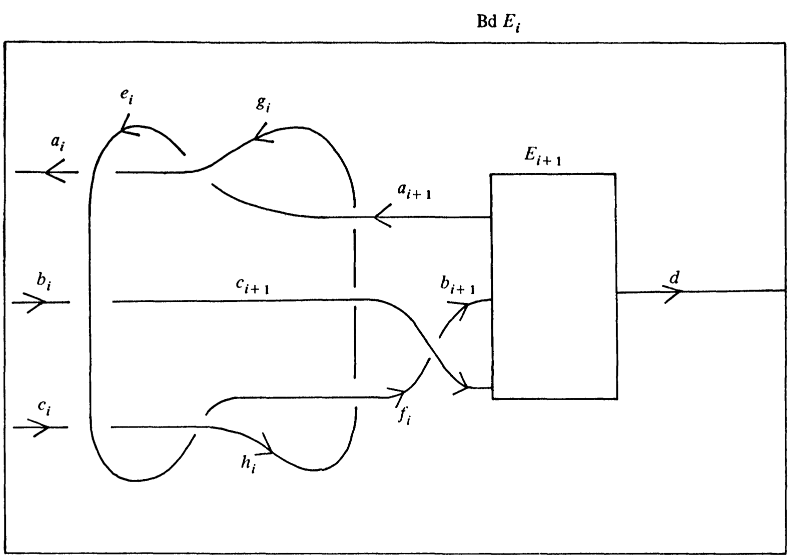
(i) *Examples* 1.1 *and* 1.1* *of* **[4]**,

(ii) *The arcs* $A_1, A_2, \cdots$ *of* **[1]**,

(iii) *Example* 5 *of* **[3]**,

(iv) *The arc* A_0 *of* **[8]**.

For, each of these arcs has an admissible sequence for which $\pi_1(E_i - E_{i+1} - k)$ is free. For (iv), for example, the section A_0, E_i, E_{i+1} is shown in the figure. Using the relations given in **[8]**, it is straightforward to



$$\pi_1(E_i - E_{i+1} - A_0) = \left\langle \begin{matrix} a_i, b_i, \\ c_i, e_i \end{matrix} : \begin{matrix} a_i e_i a_i = \\ (b_i c_i) e_i c_i e_i^{-1} (b_i c_i)^{-1} a_i e_i \end{matrix} \right\rangle$$

show that the fundamental group is the free group generated by the set $\{e_i, a_i, b_i c_i\}$.

THEOREM 2. *Let k be an arc which is not locally prime at p. Then there exists an arc k^* which is locally prime at p, and a sequence κ of prime knots, such that $k = k^* \# \kappa$. That is, every arc which is not locally prime has a prime factorisation in* Int E_0.

Let $\pi_1, \pi_2, \cdots$ be an ordering of the oriented prime tame knots, and let κ be a sequence of such knots. For each i, let $e_i(\kappa)$ denote the number of times the prime π_i occurs in κ; if π_i occurs infinitely often, set $e_i(\kappa) = \infty$.

THEOREM 3. *Let k be an arc which is not locally prime at its wild point, and let $k_1 \# \kappa_1$ and $k_2 \# \kappa_2$ be prime factorisations of k. Then k_1 and k_2 are locally equivalent, and*

(i) *$e_i(\kappa_2) = \infty$ iff $e_i(\kappa_1) = \infty$, for a given i; that is, κ_1 and κ_2 have the same set of infinitely occurring primes, and*

(ii) *if $e_i(\kappa_1)$ is finite, then $e_i(\kappa_2) = e_i(\kappa_1)$; and this is true for all but finitely many i.*

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