MARKET REPRESENTATIONS OF n-PERSON GAMES

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Communicated by Alex Rosenberg, September 22, 1973

ABSTRACT. The class of n-person games (without side payments) obtainable from a certain economic market model is characterized as the class of totally balanced games. Various market representations for a given totally balanced game are considered, and a finite commodity representation without production is obtained.

1. **Introduction.** It is the purpose of this announcement to describe some results on the problem of characterizing a certain class of n-person games which arise from considerations in economics. The details of these can be found in several different papers [2], [3], [4] appearing elsewhere. In addition, we will briefly discuss an equivalence notion for economic markets, and describe an equivalent form of one of the representations in [2].

We first establish some notation. Let $N=\{1, \cdots, n\}$ and $2^N = \{S \subset N | S \neq \varnothing \}$. For $S \in 2^N$, let $R^S = \{x \in R^n | x_i = 0 \text{ for } i \notin S\}$ and $R_+^S = R^S \cap R_+^n$, where $R_+^n = [0, \infty)^n$. If A, $B \subset R^n$ and $\alpha \in R_+$, we write $A - B = \{a - b | a \in A, b \in B\}$ and $\alpha A = \{\alpha a | a \in A\}$. For $S \subset N$, let $\chi_S \in R^n$ be given by $(\chi_S)_i = 1$ if $i \in S$, $(\chi_S)_i = 0$ if $i \notin S$.

DEFINITION 1.1. A *game* V on N is a function from 2^N to subsets of R^n, such that for each $S \in 2^N$, $V(S) = C_S - R_+^S$ where $C_S \subset R^S$ is nonempty, compact and convex.

DEFINITION 1.2. A *market* on N is a set $\{(X^i, Y^i, u_i, \omega^i) | i \in N\}$ where for each $i \in N$,

(1.2.1) X^i, Y^i are nonempty, compact, convex subsets of a real Hausdorff linear topological space E,

(1.2.2) $u_i : X^i \to R$ is concave and upper-semicontinuous, and

(1.2.3) $\omega^i \in X^i - Y^i$.

A market is a simple economic model of a production-trading situation involving n participants, called traders. The space E corresponds to the space of economic goods. The sets X^i and Y^i, called consumption and

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production sets, respectively, describe the ith trader's situation with regard to these sectors of the economy, while the function u_i describes his preferences for possible consumptions and the point ω^i specifies his position prior to engaging in the economic process. A market where production is absent (i.e. each $Y^i = \{0\}$) will be denoted by $\{(X^i, u_i, \omega^i) | i \in N\}$. A special example of this type is given by taking $E = R^m$, $X^i = I^m = [0, 1]^m$ for each i, and $\sum_{i \in N} \omega^i \leqq (1, 1, \cdots, 1)$. These will be termed *simple markets* (see [10]), and have a straightforward interpretation: each trader comes to the marketplace with an initial vector of goods ω^i. By trading with each other, the traders attempt to obtain a more preferred vector of goods (i.e. one giving a higher value of u_i for each i). For further discussion of markets see Debreu [5].

Given a market $\{(X^i, Y^i, u_i, \omega^i) | i \in N\}$ on N, one defines a game V on N, as follows: for each $S \in 2^N$,

$$V(S) = \left\{ z \in R^S \ \middle| \ z_i \leqq u_i(x^i), \ x^i \in X^i, \ i \in S; \sum_{i \in S} (x^i - \omega^i) \in \sum_{i \in S} Y^i \right\}.$$

It can be shown that V so defined is always a game, called the *market game* of the given market. In this game, $V(S)$ is essentially the set of utility outcomes for S obtainable from the market process. Any market that gives rise to a game V in this manner will be called a (market) *representation* of V.

One object of the current study is to characterize market games. In §2 we will duscuss such a characterization. In §3 we will define a transformation on markets which leaves their market games invariant. In §4 we will obtain a market representation with production for any market game, and then apply this transformation to obtain a representation without production.

Before continuing, we require one further concept. For each $S \in 2^N$, let

$$B(S) = \left\{ \delta : 2^S \to R_+ \ \middle| \ \sum_{R \ni i} \delta(R) = 1, \text{ all } i \in S \right\}.$$

If V is a game on N and $\delta \in B(S)$, let $V_\delta = \sum_{R \in 2^S} \delta(R) V(R)$. We say the game V is *totally balanced* if for each $S \in 2^N$, $V(S) = \bigcup \{V_\delta | \delta \in B(S)\}$. For any game V its cover $\bar{V}$ is the game given by $\bar{V}(S) = \bigcup \{V_\delta | \delta \in B(S)\}$. $\bar{V}$ is the smallest totally balanced game containing V (see [2]).

2. **The characterization.** The main result of this study is the following, which is a generalization of a result of Shapley and Shubik [10].

THEOREM 2.1. *A game V is a market game if and only if it is totally balanced.*

It is easily seen that market games are always totally balanced. To prove the converse, a representing market is constructed for each totally balanced game. We can obtain three different representations for a given totally balanced game (and thus an infinite number by applying [3, Lemma 2.2]). These are of the following forms:

(a) $\{(I^\infty, u_i, \omega^i)|i \in N\}$ over $E=R^\infty$ (with the Tychonoff topology, or l_∞ with the weak* topology),

(b) $\{(X^i, Y^i, u_i, \omega^i)|i \in N\}$ over $E=R^{2n}$, with $X^i=I^n \times \{0\}$, and $u_i(x, 0)=x_i$, and

(c) $\{(\hat{X}^i, \hat{u}_i, \omega^i)|i \in N\}$ over $E=R^{2n}$.

In §4, we will describe representation (b) in detail, and derive representation (c) from it. Representation (a) is derived in [2].

For a restricted class of totally balanced games, the representing market may be chosen to be a simple market. A representation by a simple market will be called a *simple representation*. A game V is said to be *polyhedral* if each $V(S)$ is a polyhedron. We then have (see [4]):

THEOREM 2.2. *A totally balanced polyhedral game always has a simple representation.*

We conjecture that every totally balanced game has a simple representation. Theorem 2.2 essentially implies that the games with simple representation are a dense subset of the closed set of totally balanced games (with respect to the Hausdorff metric on the "space of games" [6]). Thus the conjecture reduces to showing that the set of games with simple representation is closed in this sense. Further evidence for this conjecture is given by the following result (see [3]).

THEOREM 2.3. *Let $V=C-R_+^n$, where $C \subset R^n$ is compact and convex. Then $V=V(N)$ for the market game of some simple market over $R^{n(n-1)}$.*

3. **Game theoretic equivalence of markets.** Two markets will be said to be *game-theoretically equivalent* if they give the same market game. This defines an equivalence relation over markets. A related notion of equivalence has been considered by Shapley and Shubik [10]. The following construction was suggested to us by L. S. Shapley, who attributes it to T. Rader [7].

The *Rader equivalent* of a market $\{(X^i, Y^i, u_i, \omega^i)|i \in N\}$ is $\{(\hat{X}^i, \hat{u}_i, \omega^i)| i \in N\}$ where $\hat{X}^i=X^i-Y^i$ and $\hat{u}_i:\hat{X}^i \to R$ is defined for $z \in \hat{X}^i$ by $\hat{u}_i(z)= \sup\{u_i(z+y)|y \in Y^i$ and $z+y \in X^i\}$. Note that $\hat{X}^i$ is compact and convex since X^i and Y^i are, and that $\hat{u}_i$ is concave and upper-semicontinuous since u_i is. Further, it is routine to verify that the game generated by the Rader equivalent is the same as the game generated by the original market.

Hence,

THEOREM 3.1.   *The Rader equivalent of a market is a market that gener-ates the same game.*

Since the Rader equivalent of a market is a market without production it follows that every market is game-theoretically equivalent to a "trading market". Hence, game theoretically, production adds no generality to the class of markets.

4. The production representation and its Rader equivalent.   Let V be a game on N. We assume, by an obvious adaptation of Lemma 2.2 of [3], that for each $S \in 2^N$, $V(S) = C_S - R_+^S$ where C_S is a compact convex subset of $I^n \cap R^S$, and for each $i \in N$, $0 \in C_{\{i\}}$. For each $i \in N$, let $X^i = I^n \times \{0\} \subset R^{2n}$,

$$Y^i = \text{conv}\left\{\left(\bigcup_{S \in 2^N}[C_S \times \{-\chi_S\}]\right) \cup \{0\}\right\} \subset R^{2n},$$

and $\omega^i = (0, \chi_{\{i\}})$. Finally let $u_i : X^i \to R$ be given by $u_i(z, 0) = z_i$ for each $i \in N$, where $(z, 0) \in X^i$. The market $\{(X^i, Y^i, u_i, \omega^i) | i \in N\}$ is called the *market induced by V.* Roughly, the production process defined is one in which any point in C_S can be "produced" from the pooled initial resources of the traders in S. The basic idea for this construction is implicit in a con-struction made by Shapley and Shubik [10], and is explicitly used in a construction of Scarf [8] (an early unpublished version of his well-known paper [9]).

The following theorem is proved in [2].

THEOREM 4.1.   *Let $\tilde{V}$ be the market game of the market induced by V. Then $\tilde{V} = \bar{V}$.*

Since, by definition, V totally balanced implies $V = \bar{V}$, Theorem 2.1 (representation (b)) follows from Theorem 4.1. Representation (c) follows from representation (b) and Theorem 3.1. Note, however, that Theorem 3.1 applies to general markets, and thus, in deducing representation (c), we have not made use of the fact that induced markets have rather special utility functions—$u^i(x, 0) = x_i$. In particular, we would like to claim that $\hat{u}^i$ is continuous for $i \in N$. Unfortunately, we cannot.

THEOREM 4.2.   *Let V be a totally balanced game with induced market $\{(X^i, Y^i, u_i, \omega^i) | i \in N\}$ having Rader equivalent $\{(\hat{X}^i, \hat{u}_i, \omega^i) | i \in N\}$. Then $\hat{u}_i$ is continuous for each $i \in N$ only if for each $S \in 2^N$ the Pareto surface P_S of $V(S)$ is closed. (P_S is the set of maximal elements in $V(S)$ with respect to the normal partial order on R^S.)*

PROOF.   Suppose $S \in 2^N$ is such that P_S is not closed. Take $\{y^j\} \subset P_S$ such that $y^j \to y \notin P_S$. Choose $\bar{y} \in P_S$ so that $\bar{y} \geq y$ and $\bar{y}_k > y_k$ for some $k \in S$.

Put $x=0-(y, -\chi_S) \in \hat{X}^k$ and $x^j=0-(y^j, -\chi_S) \in \hat{X}^k$, $j=1, 2, \cdots$. But then $x^j \to x$, $\hat{u}_k(x^j)=0$ for each j and $\hat{u}_k(x) \geq \bar{y}_k - y_k > 0$. $\square$

An example of a Pareto surface which is not closed is given in [1], and this directly leads to a game for which the Rader equivalent of the induced market has a discontinuous $\hat{u}_i$. We remark that the converse of Theorem 3.2 is true for $n \leq 3$ but false for $n \geq 4$.

We close by making some comments concerning the set of market games on a fixed set. First, this set of games can be given the structure of a lattice by defining the meet of two games to be their pointwise intersection, and the join to be the cover of the game which is pointwise the convex hull of their union (see [2]). The set of games with simple representation is closed under the meet operation [3, Lemma 2.2]. Second, the set of market games can also be given the structure of a convex cone, i.e. it is closed under pointwise addition and nonnegative scalar multiplication. As we have already noted, this cone is closed. It contains the games with simple representation as a convex subcone [3, Lemma 2.2].

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