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GENERALIZED SUPER-PARABOLIC FUNCTIONS

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The purpose of this note is to announce results which generalize potential theory (superharmonic functions) to a broad class of parabolic operators. Many of the properties of superharmonic functions carry over to functions in this new class. Let $Q=\Omega\times(0,T)$ where $\Omega\subset E^n$ is a bounded domain and $T>0$ is a scalar. All functions will be defined on $\bar{Q}$ and will be written as functions of (x,t) with $x\in\bar{\Omega}$ and $t\in[0,T]$.

For $(x,t)\in\bar{Q}$ assume

(a) $a_{ij}(x,t)$ is a bounded, measurable function for $i,j=1,2,\cdots,n$ and assume there is a constant $\lambda>0$ such that $\sum a_{ij}(x,t)z_iz_j\geqq\lambda|z|^2$ for all $z\in E^n$ and almost all $(x,t)\in Q$.

(b) $c(x,t)\in L^q[0,T;L^p(\Omega)]$ for $n/2p+1/q<1$, $1<p,q\leqq\infty$.

(c) $b_j(x,t), d_j(x,t)\in L^q[0,T;L^p(\Omega)]$ for $j=1,\cdots,n$ and $n/2p+1/q<\frac{1}{2}$, $2<p,q\leqq\infty$.

The parabolic operator under consideration is defined by

$$Lu=u_t-\{a_{ij}(x,t)u_{,i}+d_j(x,t)u\}_{,j}-b_j(x,t)u_{,j}-c(x,t)u$$

where $u_{,j}=\partial u/\partial x_j$ and an index i or j is summed over $1\leqq i,j\leqq n$ whenever it is repeated in a product.

DEFINITION 1. $u(x,t)$ is a *weak solution of* $Lu=0$ in Q if u is locally in $L^2[0,T;H^{1,2}(\Omega)]$ and $\iint_Q[a_{ij}u_{,j}\phi_{,j}+d_j\phi_{,j}u-b_ju_{,j}\phi-cu\phi-u\phi_t]\,dx\,dt=0$ for all $\phi\in C_0^1(Q)$.

Let $\partial_pQ=\{\partial\Omega\times[0,T]\}\cup\{\Omega\times(0)\}$ denote the parabolic boundary of Q. Due to the number of definitions and results, they are stated below with no proofs.

THEOREM 1. *Let $f\in C(\partial_pQ)$ and let $u=u(x,t)$ be the weak solution of the boundary value problem*

$$Lu=0\quad on\ Q.\qquad u=f\quad on\ \partial_pQ.$$

Then, to each $(x,t)\in Q$, there corresponds a nonnegative Borel measure

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$\mu_{(x,t)}$ *on* $\partial_p Q$ *such that*

$$u(x,t) = \int_{\partial_p Q} f\, d\mu_{(x,t)} \quad on\ Q.$$

In the future, write $L(f; (x,t), Q) = \int_{\partial_p Q} f\, d\mu_{(x,t)}$.

DEFINITION 2. $u \in S_Q$ if and only if $u \in L^2[0, T; H^{1,2}(\Omega)]$ and for all $\phi \in C_0^1(Q^i)$ with $\phi \geqq 0$, $\iint_Q [a_{ij}u_{,j}\phi_{,i} + d_j\phi_{,j}u - b_j u_{,j}\phi - cu\phi - u\phi_t]\, dx\, dt \geqq 0$.

DEFINITION 3. $R_a(x_0, t_0) \equiv \{(x,t); |x_i - x_{0i}| < a, t_0 - a^2 < t \leqq t_0\}$ is called a standard rectangle based at (x_0, t_0).

DEFINITION 4. $u \in l(Q)$ if and only if

(i) $u \not\equiv +\infty$ on Q,

(ii) $u > -\infty$ on Q, and

(iii) u is lower semicontinuous on Q.

DEFINITION 5. The extended real valued Borel measurable function u defined on an open set D is

(a) *super-mean-valued* at $z \in D$ if $L(u; z, R_\delta)$ is defined and $u(z) \geqq L(u; z, R_\delta)$ for almost all δ with $\bar{R}_\delta \subset D$;

(b) *super-mean-valued* on D if it is super-mean-valued at each $z \in D$;

(c) *locally super-mean-valued at* $z \in D$ if there is a $\delta(z) > 0$ such that $\bar{R}_{\delta(z)} \subset D$ and $u(z) \geqq L(u; z, R_\delta)$ for all $\delta < \delta(z)$;

(d) *locally super-mean-valued* on D if it is locally super-mean-valued at each $z \in D$.

DEFINITION 6. $S'_Q = \{u \in l(Q); u$ is super-mean-valued on $Q\}$. $S''_Q = \{u \in l(Q)$; for any cylinder $W = C \times (a,b)$ with $\overline{W} \subset Q$, and any v with $v \in C(\overline{W})$, $Lv = 0$ on W, and $u \geqq v$ on $\partial_p W$, it follows that $u \geqq v$ on $W\}$. $S''' = \{u \in l(Q)$; u is locally super-mean-valued on $D\}$.

THEOREM 2. *$u \in S_Q$ with $u \geqq 0$ on $\partial_p Q$ implies $u \geqq 0$ on Q.*

COROLLARY. *If $c + \{d_j\}_{,j} \leqq 0$ weakly on Q, then the weak solution u of $Lu = 0$ in Q, $u = 1$ on $\partial_p Q$ satisfies $0 \leqq u(x,t) \leqq 1$ on Q.*

From now on assume $c + \{d_j\}_{,j} \leqq 0$ weakly on Q.

THEOREM 3. *Let $u \in S'''_Q$. If, for some $(x_0, t_0) \in Q$, $u(x_0, t_0) = \inf_Q u \leqq 0$, then $u(x,t) \equiv u(x_0, t_0)$ on $\Omega \times (0, t_0)$.*

THEOREM 4. $S_Q \subset S'_Q = S''_Q = S'''_Q$.

THEOREM 5. *Let $F(x)$ be convex on E^n with $F(0) \leqq 0$. If $Lu = 0$ on Q, then $-F(u) \in S'_Q$.*

THEOREM 6. *Let $F(x)$ be nondecreasing and convex on E^n with $F(0) \leqq 0$. If $-u \in S'_Q$, then $-F(u) \in S'_Q$.*

THEOREM 7. *If* $u \in S'_Q$, *and if* $u(x, t) \geqq 0$, *then there exist* t_0, t_1 *with* $0 \leqq t_0 \leqq t_1 \leqq T$ *such that*

$$\begin{aligned} u(x, t) &\equiv 0 && \text{on } \Omega \times (0, t_0), \\ 0 < u(x, t) &< +\infty && \text{on } \Omega \times (t_0, t_1), \\ u(x, t) &\equiv +\infty && \text{on } \Omega \times (t_1, T). \end{aligned}$$

THEOREM 8. *If* $u, v \in S'_Q$ *and* $c>0$, *then* (i) $cu \in S'_Q$, (ii) $u+v \in S'_Q$, (iii) $\inf(u, v) \in S'_Q$.

THEOREM 9. $u, -u \in S'_Q$ *implies* $Lu=0$ *weakly on* Q.

THEOREM 10. *Let* $u \in S'_Q$ *and let* R *be a standard rectangle with* $\bar{R} \subset Q$. *Set*

$$\begin{aligned} v(x, t) &= L(u; (x, t), R) && (x, t) \in R, \\ &= u(x, t) && (x, t) \in Q - R. \end{aligned}$$

Then $u \geqq v$ *on* Q, $Lv=0$ *on* R, *and* $v \in S'_Q$.

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