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# BOUNDARIES OF COMPLEX ANALYTIC VARIETIES

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Let $M$ be a compact, odd-dimensional submanifold of complex euclidean space $C^n$. It is perfectly natural to ask for conditions on $M$ which guarantee that it forms the boundary of a complex submanifold or, more generally, of a complex subvariety in $C^n$. Whenever $M$ does bound a subvariety $V$, then $V$ is an integral current of least mass having boundary $M$, and, moreover, $V$ is the *unique* integral current of least mass having boundary $M$ (see [3] and [7]). Thus, such conditions on $M$ could be interpreted as properties sufficient for uniqueness and regularity of the solution to Plateau's problem.

In the case that $M$ is one-dimensional, there has been a great deal of work on this question, beginning with the fundamental results of John Wermer in 1958 [**11**] (see [**5**] or [**12**] for a survey and bibliography). The purpose of this note is to announce results for manifolds of dimension greater than one. The theorems in this case differ strikingly from those of Wermer, the difference being essentially related to the appearance in several variables of "Hartog's phenomenon". The extension theorem of Bochner (see [**2**] and [**8**]) occurs as the special case where $M$ is the graph of a function defined on the boundary of an open set in $C^{n-1}$.

Detailed proofs will appear elsewhere.

Let $M$ be a compact, orientable, $(2k-1)$-manifold differentiably embedded in $C^n$. In order that $M$ be, even locally, the boundary of a $k$-dimensional complex submanifold $V$, it is necessary that, for all $z \in M$,

$$\dim_C(T_z(M) \cap iT_z(M)) = k - 1;$$

since $T_z(M) \cap iT_z(M)$ is the orthogonal complement in $T_z(V)$ of the complex line spanned by the normal vector to the boundary $M$ at $z$. The submanifold $M$ will be called *maximally complex* if the above condition holds at all points; the condition says that the tangent space to $M$ at $z$ has a complex subspace of the maximal possible dimension. (Note that the

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generic situation is where the tangent space to $M$ has the minimal possible dimension.)

THEOREM 1. *Let $M$ be a compact, orientable, $(2k-1)$-dimensional real submanifold of $\mathbf{C}^n$ of class $C^r$ with $k$, $r \geqq 2$. If $M$ is maximally complex, then $M$ is the boundary (as a current) of a uniquely determined, bounded, complex analytic variety $V$ in $\mathbf{C}^n - M$.*

The following remarks indicate that taking the boundary in the sense of currents is appropriate. First, we note that under mild conditions of convexity or density at a point, a weak boundary (in the sense of currents) becomes a strong (classical) boundary in a neighborhood of the point. Suppose $M$ is maximally complex and give $M$ the orientation induced from $V$. Then $M$ has an associated *Levi form*, an exterior $(1, 1)$ form defined on the distribution $E$ of complex subspaces of $T(M)$ as follows. Let $\omega$ be a nonvanishing one-form on $M$ such that $\omega|_E = 0$. The orientation determines $\omega$ up to multiplication by a positive function. Then $\Phi = d\omega|E$. ($\Phi$ is also defined up to multiplication by a positive function.)

LEMMA 2. *Let $M$ be of class $C^r$, $r \geqq 2$, and suppose $M$ is the boundary of a variety $V$. If at a point $x \in M$ any of the following conditions hold:*
(i) *the Levi form $\Phi$ has a positive eigenvalue,*
(ii) *$x$ lies on the boundary of the convex hull of $M$,*
(iii) $\underline{\lim}_{r \downarrow 0} \mathrm{Vol}(V \cap B(x; r))/\alpha(2k) r^{2k} < \frac{3}{2}$, *where $\alpha(2k)$ is the volume of the unit ball in $\mathbf{C}^k$,*

*then there is a neighborhood $U$ of $x$ in $\mathbf{C}^n$ such that $V \cap U$ is a regular submanifold of $U$, with $M \cap U$ as its $C^r$ manifold boundary. (In case* (i) *we must restrict to an appropriate branch of $V \cap (U - M)$.)*

The nontrivial part of the lemma follows from [7] and the work of Hans Lewy [9].

The density condition (iii) can be shown to hold on an open, dense subset of $M$.

Examples show that strong boundary regularity will not hold in general.

$M$ is called *strictly Levi pseudoconvex* if the Levi form with respect to the orientation induced by $V$ is everywhere positive definite. Under these conditions, $M$ is the border of a complex manifold with $C^r$ boundary mapped holomorphically into $\mathbf{C}^n$ with isolated, interior singularities.

DEFINITION 1. Let $M \subset \mathbf{C}^n$ be as in Theorem 1 with $r \geqq 1$. A map $f: M \to \mathbf{C}^l$ is said to be C.-R. if for each $z \in M$, the differential $df_z$ restricted to the complex subspace of $T_z M$ is complex linear (i.e., commutes with multiplication by $i$).

Equivalently, a function $f \in C^1(M)$ is C.-R. if it satisfies the tangential Cauchy-Riemann equations $\bar{\partial}_b f = 0$. The following conditions are all equivalent: a function $f \in C^1(M)$ is C.-R.; the graph map $F(z)=(z, f(z))$ is C.-R.; and the graph $G_f$ (of $f$ over $M$) is maximally complex. The next corollary generalizes the well-known extension theorem of Bochner from open sets in $\boldsymbol{C}^n$ to pieces of subvarieties in $\boldsymbol{C}^n$. Theorem 1 can be utilized since $G_f$ is maximally complex.

COROLLARY 1. *Let $M \subset \boldsymbol{C}^n$ be as in Theorem* 1 *and let $V_M$ be the complex variety with $dV_M = M$. If $f \in C^2(M)$ satisfies $\bar{\partial}_b f = 0$ on $M$ then $f$ extends to a weakly holomorphic function on $V_M$.*

COROLLARY 2. *Let $M \subset \boldsymbol{C}^n$ and $M' \subset \boldsymbol{C}^{n'}$ be as in Theorem* 1 *and let $f: M \to M'$ be a $C^2$ map which is C.-R. as a map into $\boldsymbol{C}^n$ (i.e., $df_z$ maps the complex subspace of $T_z(M)$ complex linearly into the complex subspace of $T_{f(z)}M'$). Then $f$ extends to a weakly holomorphic map $F: V_M \to V_{M'}$.*

The methods used to prove Theorem 1 can be applied to more general objects; for example, the following.

DEFINITION 2. By a *real analytic chain* of dimension $m$ in $\boldsymbol{R}^n$ we mean a $d$-closed, $m$-rectifiable current $W$ whose support is compact and consists of a finite union of analytic blocks. (See Federer [**4**, 4.2.28] or Hardt [**6**].) A point $x \in \operatorname{supp}(W)$ is a *regular* point if $x$ has a neighborhood $U$ such that $U \cap \operatorname{supp}(W)$ is a real analytic submanifold.

DEFINITION 3. A *holomorphic k-chain* in a domain $\Omega \subset \boldsymbol{C}^n$ is a $d$-closed, locally $2k$-rectifiable current $V$ such that $\operatorname{supp}(V)$ is a pure $k$-dimensional complex subvariety of $\Omega$. Every such current is of the form $\sum_i n_i[V_i]$ where $n_i \in \boldsymbol{Z}$ and $[V_i]$ corresponds to integration over a canonically oriented, irreducible $k$-dimensional subvariety.

THEOREM 2. *Let $W$ be a real analytic chain of dimension $2k-1>1$ in $\boldsymbol{C}^n$ such that* $\operatorname{supp}(W)$ *is maximally complex at every regular point. Then there exists a unique current $V$ with compact support in $\boldsymbol{C}^n$ such that $V$ restricted to $\boldsymbol{C}^n - \operatorname{supp}(W)$ is a holomorphic k-chain and $dV = W$.*

*Note.* If $\operatorname{supp}(W)$ is homologically irreducible (for example, if it is a connected submanifold), then either $V$ or $-V$ is a canonically oriented complex subvariety.

From the basic properties of Stein manifolds we immediately have the following.

THEOREM 3. *Theorems* 1 *and* 2 *remain valid with $\boldsymbol{C}^n$ replaced by any Stein manifold* (*or globally embeddable Stein space*).

This raises the question of to what extent the theorem remains true in larger spaces. In particular, what happens in manifolds obtained by removing subvarieties from complex projective $n$-space $\boldsymbol{P}^n(\boldsymbol{C})$. If the subvariety is a hypersurface, the complementary manifold is Stein and the theorems are true. More generally we have the following.

THEOREM 4. *Let $X^p$ be a pure p-dimensional subvariety of $\boldsymbol{P}^n(\boldsymbol{C})$ and assume $X^p$ is a complete intersection. Let $M$ be a compact, oriented, $(2k-1)$-dimensional submanifold of $\boldsymbol{P}^n - X^p$ (or a real analytic chain of dimension $2k-1$ in $\boldsymbol{P}^n - X^p$). If $M$ is maximally complex and $k+p>n$, then there exists a unique current $V$ with compact support in $\boldsymbol{P}^n - X^p$, which is a holomorphic chain in the complement of $M$ and has $dV=M$.*

This generalizes a theorem of H. Rossi [**10**] and Barth [**1**] which concerns the case where $M$ is already the boundary of a variety defined in a neighborhood of $X^p$.

## REFERENCES


**1.** W. Barth, *Fortsetzung meromorpher Funktionen in Tori und Komplexprojectiven Räumen*, Invent. Math. **5** (1968), 42–62. MR **36** #6646.

**2.** S. Bochner, *Analytic and meromorphic continuation by means of Green's formula*, Ann. of Math. (2) **44** (1943), 652–673. MR **5,** 116.

**3.** H. Federer, *Some theorems on integral currents*, Trans. Amer. Math. Soc. **117** (1965), 43–67. MR **29** #5984.

**4.** ———, *Geometric measure theory*, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, New York, 1969. MR **41** #1976.

**5.** T. W. Gamelin, *Polynomial approximation on thin sets*, Symposium on Several Complex Variables (Park City, Utah, 1970), Lecture Notes in Math., no. 184, Springer-Verlag, New York, 1970, pp. 50–78.

**6.** R. Hardt, *Slicing and intersection theory for chains associated with real analytic varieties*, Acta Math. **129** (1972), 75–136.

**7.** R. Harvey and B. Shiffman, *A characterization of holomorphic chains*, Ann. of Math. (to appear).

**8.** L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N.J., 1966. MR **34** #2933.

**9.** H. Lewy, *On the local character of the solution of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. of Math. (2) **64** (1956), 514–522. MR **18,** 473.

**10.** H. Rossi, *Continuation of subvarieties of projective varieties*, Amer. J. Math. **91** (1969), 565–575. MR **39** #5830.

**11.** J. Wermer, *The hull of a curve in $\boldsymbol{C}^n$*, Ann. of Math. (2) **68** (1958), 550–561. MR **20** #6536.

**12.** ———, *Banach algebras*, Markham, Chicago, Ill., 1971.



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