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DEFORMATIONS OF GROUP ACTIONS

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1. **Introduction.** The purpose of this note is to announce some results concerning the local structure of the space $A(G, M)$ of actions of a finite group G on a manifold M, endowed with the compact-open topology. We consider the questions of how nearly alike two actions must be if they are connected by a path in $A(G, M)$ and when two sufficiently close actions must be connected by a path.

In [5] R. Palais studied the space $D(G, M)$ of *differentiable* actions on a closed smooth manifold, endowed with the C^1 topology. His main result is the following (true, in fact, for any compact Lie group G):

THEOREM 1.1. *Let φ be in $D(G, M)$. Then there is a neighborhood U of φ in $D(G, M)$ and a continuous map $f: U \to \mathrm{Diffeo}(M)$ (the latter with the C^1 topology also) such that $f(\varphi) = 1_M$ and $f(\psi) * \psi = \varphi$ for all ψ in U, where "$*$" denotes the usual action of* Diffeo(M) *on $D(G, M)$ by conjugation.*

Palais draws the following corollary (see also [6]):

COROLLARY 1.2. *If φ_t, $0 \leqq t \leqq 1$, is a path in $D(G, M)$, then there is a path f_t in* Diffeo(M) *such that $f_0 = 1_M$ and $\varphi_t = f_t * \varphi_0$, for all t.*

In particular, the space $D(G, M)$ is locally contractible, close actions are equivalent, and actions connected by a path are equivalent. Smoothness (as well as compactness) is necessary for these results. Thus in what follows we restrict attention to the topological and PL categories.

The results which follow constitute part of the author's thesis, written under the direction of Professor Frank Raymond at the University of Michigan. Details will appear elsewhere.

2. **When are G-isotopic actions equivalent?** Define a *G-isotopy* to be a level-preserving action in $A(G, M \times I)$. A PL G-isotopy is a G-isotopy in the space $A_{\mathrm{PL}}(G, M \times I)$ of PL actions. The following observation provides the typical examples of close, inequivalent, G-isotopic actions. The proof is analogous to the construction of the standard "Alexander isotopy."

PROPOSITION 2.1. *Let φ be in $A(G, D^n)$, where D^n is the unit n-disk. Then φ is G-isotopic to $C(\varphi \mid S^{n-1})$, the cone action over φ restricted to the boundary sphere S^{n-1}.*

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Now let φ above denote one of the many "strange" actions on D^n, thus producing actions arbitrarily close and G-isotopic, yet inequivalent, to $C(\varphi \mid S^{n-1})$. An inductive application of Proposition 2.1 shows that any PL action on S^n is PL G-isotopic to the join of a trivial action and a fixed point free action.

PROPOSITION 2.2. *Suppose that G is a p-group, p prime, and let θ be a G-isotopy in $A(G, M \times I)$. Then the inclusion of fixed point sets* $\mathrm{Fix}(\theta \mid M \times 0) \subset \mathrm{Fix}(\theta)$ *induces an isomorphism on Z_p-homology.*

The proof is an easy application of Smith theory.

PROPOSITION 2.3. *Suppose that M is a compact* PL *manifold. Then any* PL *G-isotopy on $M \times I$ passes through only finitely many inequivalent levels.*

The proof uses elementary PL topology. It is less elementary to construct examples of a Z_2-isotopy on D^n and a smooth Z_2-isotopy on R^n, each of which passes through infinitely many different levels, distinguished by the fundamental groups of their fixed point sets.

We now describe a certain local unknottedness property which guarantees that a PL G-isotopy θ in $A_{\mathrm{PL}}(G, M \times I)$ is equivalent to the trivial G-isotopy $\theta_0 \times 1$, which at every level is the same as the action θ at the 0 level. Call θ *locally unknotted* if for each (x, t) in $M \times I$, with $t < 1$, there is a θ_t-invariant closed PL neighborhood U of x in M and an equivariant PL embedding $h:(U \times I, \theta_t \times 1) \to (M \times [t, 1], \theta)$ onto a neighborhood of (x, t) in $M \times [t, 1]$, such that $h(x, 0) = (x, t)$ for all x in U. An analogous condition is required to hold for $t = 1$. The PL G-isotopy θ is *unknotted* if there is a level-preserving equivariant PL homeomorphism $(M \times I, \theta_0 \times 1) \to (M \times I, \theta)$ whose restriction to $M \times 0$ is the inclusion.

THEOREM 2.4. *On a compact* PL *manifold a* PL *G-isotopy is unknotted if and only if it is locally unknotted.*

The proof uses an equivariant local collaring-implies-collaring theorem plus ad hoc techniques. In particular this shows that PL G-isotopic free actions on a compact manifold are PL equivalent, since by Proposition 2.2 any G-isotopy between them is free and any free PL G-isotopy is easily seen to be locally unknotted. This observation also holds in the topological category by a different proof.

The following result makes even clearer the observation that the obstructions to constructing an equivalence between PL G-isotopic actions are local in nature.

THEOREM 2.5. *Let M be a closed* PL *n-manifold and θ in $A_{\mathrm{PL}}(G, M \times I)$*

be a PL *G-isotopy. Then there is a sequence* $\varphi_1, \ldots, \varphi_r$ *of actions in* $A_{PL}(G, M)$ *such that* $\varphi_1 = \theta_0$, $\varphi_r = \theta_1$, *and* $\varphi_{i+1} = \varphi_i$ *except on the invariant disjoint union of open n-cells, for* $i = 1, \ldots, r - 1$.

This result is analogous to the result in PL topology that "isotopy implies isotopy by moves." The theorem reduces the study of PL G-isotopies to the study of actions on disks which agree on the boundary sphere.

3. **Local connectedness of** $A(G, M)$**.** The results in this section deal primarily with three special subspaces of $A(G, M)$: the space of free actions $FA(G, M)$; the space of free PL actions $FA_{PL}(G, M)$; and the space of PL actions which are free except for possible isolated fixed points $A^0_{PL}(G, M)$.

For $FA(G, M)$, using techniques of R. Edwards and R. Kirby [**2**], we recover an exact analogue of Theorem 1.1.

THEOREM 3.1. *Let M be a closed topological manifold and* φ *be in* $FA(G, M)$. *Then there is a neighborhood U of* φ *in* $FA(G, M)$ *and a continuous map* $f: U \to \text{Homeo}(M)$, *the latter with the compact-open topology, such that* $f(\varphi) = 1_M$ *and* $f(\psi) * \psi = \varphi$ *for all* ψ *in U.*

Thus close free actions are canonically equivalent. The analogous corollary also holds.

COROLLARY 3.2. *If M is a closed topological manifold and if* θ *is a G-isotopy in* $FA(G, M \times I)$, *then there is an equivariant level-preserving homeomorphism* $(M \times I, \theta_0 \times 1) \to (M \times I, \theta)$, *which when restricted to* $M \times 0$ *is the inclusion.*

Versions of these results also hold for manifolds with boundary. By using the triangulation theorems of Kirby and Siebenmann [**4**] one obtains a similar but weaker version of Theorem 3.1 in the PL category.

THEOREM 3.3. *Let G be a finite group of odd order, M be a closed*, 3-*connected*, PL *n-manifold*, $n \geqq 5$, *and* φ *be in* $FA_{PL}(G, M)$. *If* ψ *is in* $FA_{PL}(G, M)$ *and is sufficiently close to* φ, *then there is a* (*noncanonical*) PL *G-isotopy between* ψ *and* φ, *and the G-isotopy is induced from* φ *by a* PL *ambient isotopy of M.*

Close actions with fixed points require more work.

THEOREM 3.4. *Let G be a p-group, p prime, M be a compact* PL *manifold, and* φ *be in* $A_{PL}(G, M)$. *Then for any* ψ *in* $A(G, M)$ *which is sufficiently close to* φ, *there is an equivariant map* $(M, \psi) \to (M, \varphi)$ *close to the identity which induces an isomorphism of the* Z_p-*homology of the fixed point sets.*

The proof of Theorem 3.4 uses Smith theory and an equivariant embedding trick due to Palais [**5**].

Finally our best result so far concerning close PL actions with fixed points is the following:

THEOREM 3.5. *Let G be a finite group of odd order, M be a closed, 3-connected,* PL *n-manifold, $n \geqq 6$, and φ be in $A^0_{\mathrm{PL}}(G, M)$. Then any ψ in $A_{\mathrm{PL}}(G, M)$ which is sufficiently close to φ is* PL *G-isotopic to φ, topologically equivalent to φ, but not in general* PL *equivalent to φ.*

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