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A SMASH PRODUCT FOR SPECTRA

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ABSTRACT. We shall show that the smash product for pointed CW complexes induces a smash product $\wedge$ on the homotopy category of Adams's stable category with the following properties. $\wedge$ is coherently homotopy unitary (S^0), associative, and commutative, $\wedge$ commutes with suspension up to homotopy, and $\wedge$ satisfies a Kunneth formula.

Introduction. Precisely, we shall show that the homotopy category of a technical variant of Adams's stable category [1], a fraction category of CW prespectra equivalent to that of Boardman [3], [8], admits a symmetric monoidal structure in the sense of [4].

Whitehead's pairings of prespectra [9] and Kan and Whitehead's nonassociative smash product for simplicial spectra [6] were the first attempts at a smash product. Boardman gave the first homotopy associative, commutative, and unitary smash product in his stable category [3]. Adams has recently obtained a similar construction [2].

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1. **The interchange problem.** A S^k-prespectrum X consists of a sequence of pointed spaces $\{X_n \mid n \geqq 0\}$, together with inclusions $X_n \wedge S^k \to X_{n+1}$. Consider $S = \{S_n = S^{nk}\}$ as a ring with respect to the smash product of spaces. Then X is a right S-module.

Construction of a homotopy associative smash product for S^k-prespectra requires permutations π of $S^k \wedge \cdots \wedge S^k$. Since S is not strictly commutative, but only graded homotopy commutative, this requires defining suitably canonical maps of degree -1 (for k odd) and homotopies $\pi \simeq \pm\mathrm{id}$.

We avoid sign problems by using S^4-prespectra and define *canonical homotopies* H_π as follows. Make the standard identifications

$$S^{4k} \cong S^4 \wedge \cdots \wedge S^4 \cong I^4/\partial I^4 \wedge \cdots \wedge I^4/\partial I^4 \cong I^{4k}/\partial I^{4k} \cong D^{4k}/S^{4k-1}.$$

Then π simply permutes factors of $C^2 \times \cdots \times C^2$. Hence $\pi \in SU(2k)$.

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Since $SU(2k)$ is path connected and simply connected, there is a unique homotopy class of paths $[\Gamma_\pi]$ in $SU(2k)$ with $\Gamma_\pi(0) = \pi$, $\Gamma_\pi(1) = e$. Define homotopies $H_\pi : S^{4k} \wedge I^* \to S^{4k}$ by $H_\pi(s, t) = \Gamma_\pi(t)(s)$. Then $[H_\pi]$ is the required homotopy class (rel the endpoints) of canonical homotopies.

THEOREM 1. *Let π and π' be permutations of S^{4k}. Let H_π and $H_{\pi'}$ be canonical homotopies. Then $H_{\pi'}(H_\pi(s, t), t) : S^{4k} \wedge I^* \to S^{4k}$ is a canonical homotopy for $\pi'\pi$.*

Adams solves the interchange problem directly for S^1-prespectra by an argument involving special classes of paths in SO.

2. **The Adams completion.** Give S^4 the CW structure with one 4-cell and one 0-cell. Give $[0, n]$ the CW structure with 0-cells $0, 1, \ldots, n$.

DEFINITION 2. A prespectrum consists of a sequence of pointed CW complexes $\{X_n \mid n \geqq 0\}$, together with inclusions as subcomplexes $X_n \wedge S^4 \to X_{n+1}$. A strict (weak) map of prespectra $f : X \to Y$ consists of a sequence of continuous maps $\{f_n : X_n \to Y_n\}$ such that $f_{n+1} = f_n \wedge S^4$ (resp. $f_{n+1} \simeq f_n \wedge S^4$) on $X_n \wedge S^4$. Ps is the category of prespectra and strict maps.

DEFINITION 3 (ADAMS). A subspectrum $X' \subset X$ is cofinal if for each cell σ of X a sufficiently high suspension of σ is in X'.

DEFINITION 4 (ADAMS). Ad is the category of prespectra in which maps from X to Y are diagrams $X \supset X' \to Y$ in Ps with X' cofinal in X.

Formally, Ad is the right fraction category [5] of Ps in which cofinal inclusions are invertible. Morphisms $X \supset X' \xrightarrow{f} Y$ and $X \supset X'' \xrightarrow{g} Y$ are equal if $f = g$ on $X' \cap X''$.

DEFINITION 5. Let X be a prespectrum. Given a monotone unbounded sequence of nonnegative integers $\{j_n \mid n \geqq 0, j_n \leqq n\}$, define a prespectrum DX by $(DX)_n = X_{j_n} \wedge S^{4(n-j_n)}$, with inclusions $(DX)_n \wedge S^4 \to (DX)_{n+1}$ defined so that $DX \subset X$.

Then D extends to a functor (*destabilization*) on Ps, and there are natural cofinal inclusions $DX \subset X$.

There are smash products $\wedge$: Ps, CW $\to$ Ps, and $\wedge$: Ad, CW $\to$ Ad; these are defined degreewise.

DEFINITION 6. Maps $f, g : X \rightrightarrows Y$ in Ps (Ad) are homotopic if there is a map $H : X \wedge I^* \to Y$ in Ps (resp. Ad) with $H|X \wedge 0^* = f$, $H|X \wedge 1^* = g$.

Homotopy has the usual properties. Denote the resulting homotopy categories $Ht(\mathrm{Ps})$, $Ht(\mathrm{Ad})$.

3. **Smash products.** We shall define a family of smash products $\wedge$ on $Ht(\mathrm{Ad})$.

DEFINITION 7. Let X and Y be prespectra. Given a sequence of pairs of nonnegative integers $\{(i_n, j_n) \mid n \geqq 0,\ i_n + j_n = n$, and $\{i_n\}$ and $\{j_n\}$ are monotone unbounded sequences$\}$, let $X \wedge Y$ be the prespectrum with $(X \wedge Y)_n = X_{i_n} \wedge Y_{j_n}$; the required inclusions are induced from X and Y. Then $\wedge$ extends successively to bifunctors on Ps, Ad (since the smash product of cofinal inclusions is cofinal) and $Ht(\mathrm{Ad})$.

4. **Uniqueness and the symmetric monoidal structure.**

DEFINITION 8. Let X be a prespectrum. A *permutation* Π of DX consists of a sequence of maps $\Pi_n = X_{jn} \wedge \pi_n : (DX)_n = X_{jn} \wedge S^4 \wedge \cdots \wedge S^4 \to DX_n$, where each π_n is a permutation of $S^4 \wedge \cdots \wedge S^4$. If $g = \{g_n : (DX)_n \to Y_n \mid g_{n+1}$ extends $g_n \wedge S^4$ up to permutation$\}$, call g a *permutation map*.

PROPOSITION 9. *Permutation maps are weak maps, where the required homotopies $H_n : (DX)_n \wedge S^4 \wedge I^* \to Y_{n+1}$ are induced by canonical homotopies* (§1).

Also, permutation maps may be destabilized and are closed under the following *composition*: $f : DX \to Y$ and $g : D'Y \to Z$ yield $gD'(f) : D'DX \to D'Y \to Z$. Two permutation maps $f, g : DX \rightrightarrows Y$ differ by a permutation if for some permutation Π of DX, $g = f\Pi$.

THEOREM 10. *Any two smash products $\wedge$ and $\wedge'$ on $Ht(\mathrm{Ad})$ are naturally isomorphic.*

PROOF. There are natural destabilizations and permutation classes of permutation maps $D(X \wedge Y) \to X \wedge' Y$, $D'(X \wedge' Y) \to X \wedge Y$. The composites $D'D(X \wedge Y) \to X \wedge Y$ and $DD'(X \wedge' Y) \to X \wedge' Y$ differ from the respective (cofinal) inclusions by permutations. Thus it suffices to prove the following.

LEMMA 11. *A permutation commutative diagram of permutation maps $DX \to Y$ induces a commutative diagram in $Ht(\mathrm{Ad})$.*

We shall sketch a proof in §6.

THEOREM 12. *There are natural maps in $Ht(\mathrm{Ad})$,*

$$X \to X \wedge S^0 \to X \qquad (\textit{unit}),$$
$$a : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z) \qquad (\textit{associativity}),$$
$$c : X \wedge Y \to Y \wedge X \qquad (\textit{commutativity}),$$

which form a symmetric monoidal category.

PROOF. There are natural destabilizations and permutation classes of permutation maps $DX \to X \wedge S^0 \to X$,

$$a' : D(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z),$$

and

$$c':D(X \wedge Y) \to Y \wedge X.$$

By Lemma 11, it suffices to obtain the coherency diagrams [4] as permutation-commutative diagrams of permutation maps involving destabilizations. For example, the diagram for coherency of associativity is

$$\begin{tikzcd}
D^3(((W \wedge X) \wedge Y) \wedge Z) \arrow[r, "D^2(a')"] \arrow[d, "(a' \wedge Z)_*"] & D^2((W \wedge X) \wedge (Y \wedge Z)) \arrow[r, "D(a')"] & D(W \wedge (X \wedge (Y \wedge Z))) \arrow[d, "\text{cofinal}"] \\
D^2((W \wedge (X \wedge Y)) \wedge Z) \arrow[r, "D(a')"] & D(W \wedge ((X \wedge Y) \wedge Z)) \arrow[r, "(W \wedge a')^*"] & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

where D is used generically, and $(a' \wedge Z)_*$ and $(W \wedge a')_*$ are representatives of permutation classes of permutation maps induced by a'.

These diagrams are readily obtained.

5. **Telescopes.** We sketch the main properties of the following telescope construction.

DEFINITION 13. For a prespectrum X, let Tel X be the prespectrum with $(\text{Tel } X)_n = \bigcup_{j=0}^n (X_j \wedge S^{4(n-j)} \wedge [j, n]^*)$, the iterated mapping cylinder of $X_0 \wedge S^{4n} \to \cdots \to X_{n-1} \wedge S^4 \to X_n$, together with the induced inclusions.

Then Tel may be extended to a functor, and there are natural projections $p_X:\text{Tel } X \to X$.

PROPOSITION 14. *p_X admits a homotopy inverse s_X.*

PROOF. To define s_X, show that Tel X and X are strong deformation retracts of the prespectrum Y with $Y_n = X_n \wedge [0, n]^*$ such that p_X is the composite Tel $X \to Y \to X$. □

PROPOSITION 15. *A weak map $f:X \to Y$, together with a family of homotopies for f, $\{H_n:X_n \wedge S^4 \wedge I^* \to Y_{n+1}$ from $f_n \wedge S^4$ to $f_{n+1} \mid X_n \wedge S^4\}$, induces strict maps $\phi:\text{Tel } X \to \text{Tel } Y$ and $F:X \to Y$.*

PROOF. Let $\phi_0 = f_0$, and for $n \geqq 0$, define ϕ_{n+1} by

$$\begin{aligned}
\phi_{n+1}(x, t) &= (\phi_n \wedge S^4)(x, t) && \text{for } t \leqq n, \\
&= ((f_n \wedge S^4)(x), 2t - n) && \text{for } n \leqq t \leqq n + \tfrac{1}{2}, \\
&= (H_n(x, 2t - 2n - 1), n + 1) && \text{for } n + \tfrac{1}{2} \leqq t \leqq n + 1.
\end{aligned}$$

Also, let $F = p_Y \phi s_X$. □

PROPOSITION 16. *Let $\{H_n\}$ and $\{H_n'\}$ be homotopies for a weak map $f:X \to Y$, and assume that $H_n \simeq H_n'$* rel *the endpoints for all n. Then*

$(f, \{H_n\})$ and $(f, \{H'_n\})$ induce homotopic strict maps Tel $X \to$ Tel Y *and* $X \to Y$.

PROPOSITION 17. *Let $f': X \to Y$ and $f'': Y \to Z$ be weak maps with homotopies $\{H'_n\}$ and $\{H''_n\}$, respectively. Then there are composed homotopies $\{H_n\}$ for $f''f'$ such that for the respective induced strict maps $F': X \to Y$, $F'': Y \to Z$, and $F: X \to Z$, $F \simeq F''F'$.*

6. **Proof of Lemma 11.** Given a permutation-commutative square in which each map and composite (see §4) is a permutation map,

$$\begin{array}{ccc} DW & \xrightarrow{f'} & D'X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f''} \\ D''Y & \xrightarrow{g''} & Z, \end{array}$$

replace each map by a strict map to obtain a commutative square in $Ht(\mathrm{Ps})$ (by Theorem 1, and §5).

$$\begin{array}{ccccc} DW & \xrightarrow{P} & DW & \xrightarrow{F'} & D'X \\ \downarrow{\scriptstyle G'} & & & & \downarrow{\scriptstyle F''} \\ D''Y & & \xrightarrow{\quad G'' \quad} & & Z \end{array}$$

Here P is induced by a permutation Π of DW. Since, for all k, $SU(2k)$ is simply connected, we may "pull back" canonical homotopies to obtain a homotopy $P \simeq DW$. This completes the proof, since cofinal inclusions are inverted in $Ht(\mathrm{Ad})$.

7. **Further properties.** $\wedge$ commutes with the suspension

$$? \wedge S^1 : Ht(\mathrm{Ad}) \to Ht(\mathrm{Ad}),$$

$\wedge$ satisfies a Kunneth formula for stable integral homology (see [**6**]), and $\wedge$ is weakly universal for pairings (see [**9**], [**6**]). We conjecture the existence of an internal mapping functor adjoint to $\wedge$ using a suitable category of permutation maps and methods of Quillen [**7**].

ADDED IN PROOF. This follow's from Brown's Theorem by Heller [Trans. Amer. Math. Soc. **147** (1970), 573–602].

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