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SPLITTING OBSTRUCTIONS FOR HERMITIAN FORMS AND MANIFOLDS WITH $Z_2 \subset \pi_1$

BY SYLVAIN E. CAPPELL[1]

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Theorem 1 of this announcement constructs explicit algebraic counterexamples to the general conjecture (see for example [W, p. 138]) that groups of Hermitian forms satisfy a sum formula for free products. This conjecture was verified when the relevant groups have no 2-torsion by proving equivalent codimension one splitting theorems for manifolds. In part II of this note, see especially Theorem 7, the failure of the general algebraic conjecture leads to examples of nonsplittable manifolds with $Z_2 \subset \pi_1$. Some of the geometric splitting obstructions occur as differences of Arf-Kervaire invariants of base spaces and covering spaces.

I. Let (G, ω) be a group G equipped with a homomorphism $\omega: G \to Z_2$. $L_n(G, \omega)$ denotes the Wall surgery obstruction group to the simple homotopy equivalence problem for manifolds with fundamental group G and orientation homomorphism ω [W]. For $n = 2k$ these are Grothendieck groups of $(-1)^k$ Hermitian forms over the integral group ring $Z[G]$; for n odd, these are abelian quotients of unitary groups over $Z[G]$. When ω is trivial, write simply $L_n(G) = L_n(G, \omega)$, and for the reduced group write $\tilde{L}_n(G)$, where $L_n(G) = \tilde{L}_n(G) \oplus L_n(0)$. Write Z (resp. Z_2) for the integers (modulo 2).

The conjecture referred to above is that $\tilde{L}_n(G_1 * G_2) = \tilde{L}_n(G_1) \oplus \tilde{L}_n(G_2)$. For G_1 and G_2 finitely presented and *without* elements of order 2, this was proved first by R. Lee for n even [L], and for all n by the author as a special case of a general result on surgery groups of amalgamated free products [C1] [C2] [C3]. For $n = 4k$, the author proved the above conjecture for G_1 and G_2 finitely presented groups [C2] [C3]. However, Theorem 1 limits the possible further extensions of this. Note that for $G = Z$ Theorem 1 provides a counterexample to a theorem of [M, p. 676].

THEOREM 1. *Let G be a nontrivial cyclic group. Then*

$$Z_2 \subset L_{4k+2}(Z_2 * G)/L_{4k+2}(Z_2) + L_{4k+2}(G).$$

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PROOF. Let t denote a generator of G and $\alpha \in Z_2$ with $\alpha \neq 0$. Let (M, λ, μ) be the Hermitian form over $Z[Z_2 * G]$ defined by:

(i) M is a free module on two generators $\{e, f\}$

(ii) $\lambda(e, e) = \lambda(f, f) = 0$, $\lambda(e, f) = 1$

(iii) $\mu(e) = \alpha$, $\mu(f) = t\alpha t^{-1}$.

Write x for the element of $L_{4k+2}(Z_2 * G)$ represented by (M, λ, μ); we show $x \notin \mathrm{Image}(L_{4k+2}(Z_2) \oplus L_{4k+2}(G)) \to L_{4k+2}(Z_2 * G)$.

Let y denote the image of the generator of $L_{4k+2}(0)$ in $L_{4k+2}(Z_2 * G)$. The proof will be completed by showing that x and y have the same images under the map $L_{4k+2}(Z_2 * G) \to L_{4k+2}(Z_2) \oplus L_{4k+2}(G)$, but $x \neq y$. Trivially, the images of x and y in $L_{4k+2}(Z_2) \cong L_{4k+2}(0) \cong Z_2$ [**W**] have nonzero Arf-invariants and so are equal. Moreover, the image of (M, λ, μ) in $L_{4k+2}(G)$ is obviously in $\mathrm{Image}(L_{4k+2}(0) \to L_{4k+2}(G))$ and, as it has nonzero Arf-invariant, image(x) = image(y) in $L_{4k+2}(G)$.

Lastly, we show $x \neq y$ by constructing a homomorphism which is zero on x but nonzero on y. Choose $\varphi: Z_2 * G \to D$, a homomorphism to D a dihedral group of order $2p$, p some *odd* number greater than 1, with $\varphi(\alpha) \neq \varepsilon$, $\varphi(t) \neq \varepsilon$, $\varphi(t) \neq \varphi(\alpha)$, ε the identity element of D; it is trivial to explicitly construct such a φ. Choose $H \subset D$ with $H \cong Z_2$ and consider the composite map

$$L_{4k+2}(Z_2 * G) \xrightarrow{\varphi_*} L_{4k+2}(D) \xrightarrow{\mathrm{tr}} L_{4k+2}(H) \cong L_{4k+2}(0) \cong Z_2$$

where tr is the transfer homomorphism [**W**, p. 242]. As p is odd, $\mathrm{tr}\ \varphi_*(y) \neq 0$. However, $\mathrm{tr}\ \varphi_*(x) = \mathrm{tr}\ \varphi_*(M, \lambda, \mu) = (M', \lambda', \mu')$. Here M' is the $Z[H]$ module $M \otimes_{Z[Z_2 * G]} Z[D]$ with a $Z[H]$ basis for M' given by $\{e \otimes v^i, f \otimes v^i\}_{0 \leqq i < p}$, v an element of order p in D; $\lambda'(e \otimes v^i, f \otimes v^j) = 0$ if $i \neq j$ and 1 if $i = j$, $0 \leqq i < p$, $0 \leqq j < p$, and $\mu'(e \otimes v^i) = 0$ if $v^{-i}\varphi(\alpha)v^i \notin H$, $\mu'(f \otimes v^i) = 0$ if $v^{-i}\varphi(t\alpha t^{-1})v^i \notin H$. But in D as

$$v^{-i}\varphi(\alpha)v^i \neq v^{-i}\varphi(t\alpha t^{-1})v^i,$$

they are not both in H; thus the Arf invariant of (M', λ', μ') is 0.

For $G = Z$ and setting $x_i = (M, \lambda, \mu_i)$ where M and λ are as above and $\mu_i(e) = \alpha$, $\mu_i(f) = t^i\alpha t^{-i}$ the proof of Theorem 1 can be easily modified by using maps of $Z * Z_2$ to various dihedral groups to show that the subgroup generated by $\{x_i\}$ is not finitely generated.

THEOREM 2. *$L_{4k+2}(Z * Z_2)$ is not finitely generated.*

From this, it is easy to construct finitely presented groups G with, for all n, $L_n(G)$ not finitely generated.

Using geometric methods, $L_{4k+2}(G_1 * G_2)/L_{4k+2}(G_1) + L_{4k+2}(G_2)$ can be computed in terms of groups of unitary nilpotent objects [**C4**].

Theorem 1 gives nonvanishing results for these groups which lead to the following:

THEOREM 3. *Let G_1 and G_2 be nontrivial finitely presented groups and assume that at least one of them has an element of order* 2. *Then* $\tilde{L}_{4k+2}(G_1 * G_2) \neq \tilde{L}_{4k+2}(G_1) \oplus \tilde{L}_{4k+2}(G_2)$.

Let Z_2^- denote the pair (Z_2, ω) where $\omega: Z_2 \to Z_2$ is the identity; similarily $Z_2^- * Z_2^-$ denotes the pair $(Z_2 * Z_2, \omega)$ with $\omega: Z_2 * Z_2 \to Z_2$ restricting to the nontrivial homomorphism on both given copies of Z_2. Extending our methods to this case, we get:

THEOREM 4. $L_{4k+2}(0) \cong L_{4k+2}(Z_2^- * Z_2^-)$; *but*

$$Z_2 \subset L_{4k}(Z_2^- * Z_2^-)/L_{4k}(Z_2^-) \oplus L_{4k}(Z_2^-).$$

In fact, $Z_2 \cong L_{4k+2}(Z_2^- * Z_2^- * Z_2^- * \cdots * Z_2^-)$.

The analogues of all the above results for the Wall groups $L_n^h(G)$, the surgery obstruction group for the homotopy equivalence problem, are also true. However, for surgery groups $L_n(G; R)$ of the group-ring $R[G]$, $Z[\frac{1}{2}] \subset R \subset Q$, we have $\tilde{L}_n(G_1 * G_2; R) = \tilde{L}_n(G_1; R) \oplus \tilde{L}_n(G_2; R)$. The results of this note, including the computation of groups of unitary nilpotent objects, are special cases of results on surgery groups of amalgamated free products.

II. **Splitting obstructions for manifolds with** $Z_2 \subset \pi_1$. Theorems 1 and 3 imply realization theorems for codimension 1 splitting obstructions for oriented $4k + 1$ dimensional manifolds with fundamental group $G_1 * G_2$ where $G_1 \neq 0$, $G_2 \neq 0$ and $Z_2 \subset G_1 * G_2$. As an application, consider the following problem: Is every manifold, homotopy equivalent to a connected sum of manifolds, itself a nontrivial connected sum? Write # for connected sums and say the (differentiable) manifold W is not a nontrivial connected sum if $W = P \# Q$ implies P or Q is a (homotopy) sphere.

THEOREM 5. *Let Y be a closed manifold* (*or Poincaré complex*) *of dimension* $n \geqq 5$ *and W a closed P.l. manifold*, $f: W \to Y$ *a homotopy equivalence. If Y is a connected sum*, $Y = Y_1 \# Y_2$, *then if either*

(i) $\pi_1(Y_1) = 0$, *or*

(ii) $\pi_1(Y)$ *has no elements of order* 2, *or*

(iii) $n = 2k + 1$ *and for each*[2] *element* $g \in \pi_1(Y)$ *with* $g \neq 1$, $g^2 = 1$, *we have* $\omega(g) = (-1)^{k+1}$ *for ω the orientation homomorphism of Y*, $\omega: \pi_1(Y) \to Z_2 = \{\pm 1\}$.

[2] It suffices to verify the condition for $g \in \pi_1(Y_1)$ and $g \in \pi_1(Y_2)$.

(*Note. This includes the case k odd, Y orientable*); *then*
(*) $W = W_1 \# W_2$, W_1 *and* W_2 *P.l. manifolds with* $f_i: W_i \to Y_i$ *homotopy equivalence and* $f = f_1 \# f_2$.

For $n \geqq 6$ and $\pi_1(Y) = 0$ the above is due to Browder [**B**]; for $n \geqq 6$ case (i) is due to Wall [**W**]. For n odd and greater than 5 and $\pi_1 Y$ without elements of order 2 this was proved by R. Lee [**L**]. For the general case see [**C1**] [**C2**] [**C3**] [**C4**].

The necessity of a restriction on $\pi_1 Y$ is shown by:

THEOREM 6. *Let Y be an n-dimensional connected sum* $Y = Y_1 \# Y_2$ *of closed P.l. manifolds,* $n \geqq 5$. *If* $\pi_1(Y_1) \neq 0$, $\pi_1(Y_2) \neq 0$; *and*
(i) $n = 4k + 1$, *Y orientable and* $\pi_1(Y)$ *has an element of order* 2 *or more generally*
(ii) $n = 4k + 1$ *and* $\exists g \in \pi_1(Y)$ *with* $g \neq 1$, $g^2 = 1$ *and letting* $\omega: \pi_1 Y \to Z_2$ *be the orientation homomorphism* $\omega(g) \neq -1$ *or*
(iii) $n = 4k + 3$ *and*

$$\exists g \in \pi_1 Y \text{ with } g^2 = 1 \text{ and } \omega(g) = -1,$$

there exists a closed manifold W, with $f: W \to Y$ *a homotopy equivalence for which there do not exist* W_1, W_2, f_1, f_2 *satisfying* (*).

The following sharp example indicates the complications arising in the classification of manifolds with $Z_2 \subset \pi_1$.

THEOREM 7. *There is a closed differentiable manifold W, simple homotopy equivalent to* $RP^{4k+1} \# RP^{4k+1}$, $k \geqq 1$, *which is not as a differentiable, P.l. or topological manifold a nontrivial connected sum.*

W may be chosen to be tangentially homotopy equivalent and even normally cobordant to $RP^{4k+1} \# RP^{4k+1}$. Theorem 7 contrasts with the situation for 3-dimensional manifolds.

OUTLINE OF THE PROOF OF THEOREM 7. (i) Realize the element of $L_{4k+2}(Z_2 * Z_2)$ constructed in the proof of Theorem 1 by a normal cobordism on $RP^{4k+1} \# RP^{4k+1}$. This gives an unsplittable homotopy equivalence $f: W \to RP^{4k+1} \# RP^{4k+1}$.

(ii) If $W = P \# Q$, as the universal cover of $RP^{4k+1} \# RP^{4k+1}$ is $S^{4k} \times R$ Mayer-Vietoris sequences show the universal covers of P and Q are highly connected. Thus, if P and Q are not spheres, as $\pi_1 W = (\pi_1 P) * (\pi_1 Q)$, P and Q are homotopy equivalent to RP^{4k+1}. Therefore some homotopy equivalence $g: W \to RP^{4k+1} \# RP^{4k+1}$ would be splittable.

(iii) But, by computing the auto-homotopy equivalences of $RP^{4k+1} \# RP^{4k+1}$ and checking that they all split, it follows that f would also split.

Obviously $L_n(G_1 * G_2)/L_n(G_1) + L_n(G_2)$ is a direct summand of $L_n(G_1 * G_2)$. Recall the action [W] of $L_{n+1}(G)$ on $\mathscr{S}_H(M)$ for M an n-dimensional orientable manifold, $n \geqq 5$, with $\pi_1 M = G$; here for $H = 0$ (resp. *P.L.*, Top) $\mathscr{S}_H(M)$ is the set of simple homotopy smoothings (resp: triangulations, "topologizations") of M. For $G = G_1 * G_2$ this restricts to an action of $L_{n+1}(G_1 * G_2)/L_{n+1}(G_1) + L_{n+1}(G_2)$ on $\mathscr{S}_H(M)$.

THEOREM 8. *Let M be an orientable closed manifold of dimension $n \geqq 5$ with $\pi_1 M = G_1 * G_2$. Then the action of*

$$L_{n+1}(G_1 * G_2)/L_{n+1}(G_1) + L_{n+1}(G_2)$$

on $\mathscr{S}_H(M)$ is free.

Thus Theorem 8 has content only if $n \neq 4k - 1$ and $Z_2 \subset G_1 * G_2$. It is a special case of a general result on the action of $L_n(G_1 *_H G_2)$ when $G_1 *_H G_2$ does not satisfy the hypothesis of the author's codimension one splitting theorem [C1] [C2] [C3] [C4].

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT, ISRAEL

DEPARTMENT OF MATHEMATICS, INSTITUT DES HAUTES ETUDES SCIENTIFIQUES, BURES-SUR-YVETTE, FRANCE