ON SUBSPACES OF SEPARABLE NORM IDEALS

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1. Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the space of all bounded linear operators on $\mathscr{H}$. In the study of operators on Hilbert space and its many applications, particularly in mathematical physics, certain ideals of the ring $\mathscr{L}(\mathscr{H})$ are of fundamental importance. These are the ones called *norm ideals* by Schatten [11] and *s.n. ideals* (*symmetric norm ideals*) by Gohberg and Krein [5]. Examples of such ideals are the space $K(\mathscr{H})$ of compact operators, the space $N(\mathscr{H})$ of nuclear (or trace-class) operators, the Hilbert-Schmidt operators and their natural extensions, the classes C_p $(1 \leq p < +\infty)$ [3], [9], and the rather recently introduced classes C_Ω^0 and C_ω of Gohberg and Krein [5], [8]. The last are already of considerable importance, arising naturally in the study of the abstract triangular integral and in questions concerning the completeness of the root vectors of an operator [1], [6], [7], [8], yet have barely been studied in any depth. In fact, though all of the ideals mentioned above are separable Banach spaces, almost no work concerning their subspace structure has been done (one exception is the fine study of the ideals C_p by McCarthy [9]).

The purpose of this paper is to begin a study of the subspaces of separable norm ideals (by subspace we always mean a *closed* linear submanifold). In particular, we study the spaces $K(\mathscr{H})$, $N(\mathscr{H})$, C_Ω^0, and C_ω from this standpoint. We refer the reader to [5] for the necessary background information and notation concerning s.n. ideals. Details of the proofs will appear elsewhere.

2. **The ideals** $K(\mathscr{H})$ **and** C_Ω^0. Though the ideals $K(\mathscr{H})$ and C_Ω^0 are dissimilar in many respects, their subspace structures are very much alike.

THEOREM 1. *Let E be a closed subspace of $K(\mathscr{H})$. Then either E is isomorphic to $\mathscr{H}$ or it contains a subspace isomorphic to c_0. If E is isomorphic to $\mathscr{H}$, then it is complemented in $\mathscr{L}(\mathscr{H})$.*

The idea of the proof of Theorem 1 is as follows: $K(\mathscr{H}) = \mathscr{H}^* \otimes_\alpha \mathscr{H}$ for a uniform crossnorm α (in fact $\alpha = \lambda$, the "least" crossnorm [11]). Moreover, if (ϕ_i) is an orthonormal basis for $\mathscr{H}$, then $(\phi_i \otimes \phi_j)$ is a basis

for $\mathscr{H}^* \otimes_\alpha \mathscr{H}$ [4]. If E is nonreflexive then it cannot be isomorphically embedded in the finite direct sum of Hilbert spaces. Consequently there exists a sequence (T_i) in E with $\|T_i\| = 1$ for all i such that $i : \|T_i\phi_n\| + \|T_i^*\phi_n\|$ $\to 0$ for each $n = 1, 2, \ldots$. Using this fact, stability theorems from basis theory [12], and the definition of the norm in $K(\mathscr{H})$, one can show that a subsequence of (T_i) is a basic sequence similar to the unit vector basis of c_0, implying $c_0 \subset E$.

If E is reflexive then no such sequence (T_i) can exist in E and so E embeds in a finite direct sum of Hilbert spaces and hence is itself isomorphic to a Hilbert space. The embedding extends to a continuous mapping of $\mathscr{L}(\mathscr{H})$ into the direct sum and from this one obtains the desired projection of $\mathscr{L}(\mathscr{H})$ onto E.

Having proved Theorem 1, we can prove that a similar situation holds for C_Ω^0.

THEOREM 2. *Let E be a closed subspace of C_Ω^0. Then either E is isomorphic to $\mathscr{H}$ or E contains c_0. If E is isomorphic to $\mathscr{H}$ then E is closed in $\mathscr{L}(\mathscr{H})$ and moreover is complemented there. In particular E is complemented in C_Ω^0.*

To prove Theorem 2 we consider the injection $i : E \to K(\mathscr{H})$. If i is an isomorphism then the theorem follows from Theorem 1. If i is not an isomorphism, then there is a sequence (T_i) in E with $\|T_i\|_\Omega = 1$ for all i but for which $\|T_i\| \to 0$. Using this fact, the stability theorem, and a rather subtle argument based on the properties of the norm on C_Ω^0, one can show that a subsequence of (T_i) is again basic and similar to the unit vector basis of c_0. If E is isomorphic to $\mathscr{H}$, then we must have E closed in $K(\mathscr{H})$ and the second part of the theorem follows from Theorem 1.

3. **The ideals $N(\mathscr{H})$ and C_ω.** Since $N(\mathscr{H}) = K(\mathscr{H})^*$ [11] and $C_\omega = (C_\Omega^0)^*$ [5], the results of this section are not entirely surprising in view of Theorems 1 and 2. However just as, in general, a knowledge of the subspaces of a Banach space X tells one little about the subspaces of X^*, so too here we cannot appeal to Theorems 1 and 2 to give the results directly.

THEOREM 3. *Let E be a closed subspace of $N(\mathscr{H})$. Then E is either isomorphic to $\mathscr{H}$ or E contains a subspace isomorphic to l^1. Moreover if E is isomorphic to $\mathscr{H}$ then E is closed in $\mathscr{L}(\mathscr{H})$ and complemented there.*

To prove Theorem 3 we consider the injection $i : E \to HS(\mathscr{H})$ (the Hilbert-Schmidt operators on $\mathscr{H}$). If E is not isomorphic to $\mathscr{H}$ then i is not an isomorphism so there exists a sequence (T_i) in E with $\|T_i\|_N = 1$ for all i but for which $\|T_i\|_{HS} \to 0$. Now essentially the same argument as was used in Theorem 1 (the only difference being that the nuclear norm determines that the subsequence be similar to (e_i) in l^1 rather than in c_0) establishes the first part of the theorem.

If E is isomorphic to $\mathscr{H}$ then the injection of E into $K(\mathscr{H})$ must also be an isomorphism and an application of Theorem 1 proves the second part of the theorem.

Again, a similar result holds for the ideals C_ω.

THEOREM 4. *Let E be a closed subspace of C_ω. Then either E is isomorphic to $\mathscr{H}$ or E contains a subspace isomorphic to l^1.*

The bulk of the proof is similar to that of the previous theorems. However here we need to also prove the following result (which, incidentally, has as a trivial corollary the result in [10]):

PROPOSITION 5. *Let λ_ω be the sequence space associated with C_ω(i.e., $(a_i) \in \lambda_\omega \Leftrightarrow \sum a_i \phi_i \otimes \psi_i \in C_\omega$). Then any closed infinite-dimensional subspace of λ_ω contains l^1.*

Finally, we remark that the methods employed in this paper are by no means limited to the investigation of spaces of operators on a Hilbert space. For example, these methods also yield the following.

THEOREM 6. *If E is a nonreflexive subspace of $K(L^p[0,1], L^s[0,1])$ $(1 < p \leqq 2, 2 \leqq s < +\infty)$ then E contains c_0.*

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