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ON HOLOMORPHIC FAMILIES OF POINTED RIEMANN SURFACES

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According to a theorem of A. Grothendieck [**4**] the Teichmüller space of a closed Riemann surface of genus $p \geqq 2$ is the universal parameter space for holomorphic families of marked Riemann surfaces of genus p. In this note we offer a corresponding description for every finite-dimensional Teichmüller space $T(p, n)$ and discuss the universal families $\pi: V(p, n) \to T(p, n)$. Detailed proofs will be given elsewhere.

1. **The space** $T(p, n)$. Let X be the smooth (C^∞) oriented closed surface of genus $p \geqq 0$, and let $x_1, x_2, \ldots$ be a sequence of distinct points on X. Set $X_0 = X$, $X_n = X \backslash \{x_1, \ldots, x_n\}$, $n \geqq 1$. Let $\mathrm{Diff}^+ X$ be the group of orientation preserving diffeomorphisms of X, with the C^∞ topology. We define the subgroups

$$\mathrm{Diff}^+ (X, n) = \{f \in \mathrm{Diff}^+ X\,;\, f(X_n) = X_n\},$$

$$G_n = \text{the path component of the identity in } \mathrm{Diff}^+ (X, n).$$

Next we form the space M of smooth conformal structures (= complex structures) on X, again with C^∞ topology. $\mathrm{Diff}^+ X$ acts on M from the right by pullback. If the inequality

$$2p - 2 + n > 0 \tag{1}$$

holds, then the group G_n acts freely, continuously, and properly (see [**3**]) with local sections, and we have a principal G_n-fibre bundle. The base space M/G_n of this bundle is, by definition, the Teichmüller space $T(p, n)$. It is well known that $T(p, n)$ has a natural complex structure and can be imbedded in C^d as a bounded open contractible domain of holomorphy [**2**], $d = 3p - 3 + n$.

2. n**-pointed families.** Suppose the integers $p, n \geqq 0$ satisfy (1). An n-pointed family (of closed Riemann surfaces of genus p) consists of a pair of complex manifolds V and B, a holomorphic map $\pi: V \to B$, and n holomorphic sections $s_j: B \to V$ such that

(i) π is a proper submersion,

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(ii) $\pi^{-1}(t)$ is diffeomorphic to the closed surface X of genus p, for all t in B,

(iii) the sections $s_1, \ldots, s_n$ are disjoint (i.e., $s_j(t) \neq s_k(t)$ for all t in B if $j \neq k$).

Given the n-pointed family $\pi : V \to B$, set

$$V' = V \setminus \bigcup_{j=1}^{n} \text{range } s_j.$$

The restriction of π maps V' onto B, and $\pi : V' \to B$ is a smooth fibre bundle with fibre X_n and structure group $\mathrm{Diff}^+(X, n)$. If the structure group of that bundle is reduced to the subgroup G_n, we say that the family $\pi : V \to B$ is *marked*. In other words, an n-pointed family is marked by choosing a homotopy basis on each "punctured fibre" $\pi^{-1}(t) \cap V'$ in a manner that depends continuously on t.

A map of marked (n-pointed) families is by definition a pair of holomorphic maps $f : V_1 \to V_2$ and $g : B_1 \to B_2$ such that $f(V_1') = V_2'$ and (f', g) is a map of G_n-bundles, where $f' = f \mid V_1'$.

THEOREM 1. *There is a marked n-pointed family $\pi : V(p, n) \to T(p, n)$ such that, for every marked n-pointed family $\pi_1 : V_1 \to B_1$, there is a unique map of marked families*

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V(p,n) \\ \pi_1 \downarrow & & \pi \downarrow \\ B_1 & \xrightarrow{g} & T(p,n). \end{array}$$

Of course the universal property described in Theorem 1 uniquely determines both $V(p, n)$ and $T(p, n)$ as complex manifolds. For $n = 0$, Theorem 1 reduces to Grothendieck's theorem [**4**]. The general case is proved by the same method. Topologically, $\pi : V(p, n) \to T(p, n)$ is the G_n-bundle with fibre X associated to the principal G_n-bundle $M \to T(p, n) = M/G_n$. The cross-sections of π are determined by the points $x_1, \ldots, x_n$ on X (which are fixed by G_n), and $\pi : V(p, n)' \to T(p, n)$ is the associated bundle with fibre X_n. The "punctured" fibre space $V(p, n)'$ is more familiar, and perhaps more natural, than $V(p, n)$. Bers has shown [**1**] that $T(p, n + 1)$ can be interpreted in a natural way as the holomorphic universal covering space of $V(p, n)'$.

3. **The modular group.** Since the group $\mathrm{Diff}^+(X, n)$ acts on M, and G_n is normal in $\mathrm{Diff}^+(X, n)$ the quotient group $\Gamma(p, n)$ acts on $T(p, n)$. $\Gamma(p, n)$ is called the (Teichmüller) modular group. This group does not always act effectively on $T(p, n)$; however, it acts also on the fibre space $V(p, n)$ and there it acts very effectively.

THEOREM 2. *$\Gamma(p, n)$ acts on $V(p, n)$ and $T(p, n)$ as a group of holomorphic* *automorphisms satisfying*

$$\pi(v \cdot \gamma) = \pi(v) \cdot \gamma \quad \textit{for all } v \in V(p, n),\ \gamma \in \Gamma(p, n).$$

Further, if $v \cdot \gamma = v$ *for all* v *in some fixed fibre* $\pi^{-1}(t_0)$, *then* $\gamma = \mathrm{id}$ *in* $\Gamma(p, n)$.

EXAMPLE. The modular group $\Gamma(2, 0)$ has in its center one nontrivial element γ, of order two. γ fixes every point of $T(2, 0)$ and therefore maps each fibre $\pi^{-1}(t)$ of $V(2, 0)$ onto itself. Each fibre is a closed Riemann surface of genus two, hence hyperelliptic, and γ on each fibre is the hyperelliptic involution. Let $\Gamma_0 = \{\gamma, \mathrm{id}\}$ be the center of $\Gamma(2, 0)$. Then $T(2, 0)/\Gamma_0 = T(2, 0) \cong T(0, 6)$, and $V(2, 0)/\Gamma_0 \cong V(0, 6)$. The six cross-sections of $\pi: V(0, 6) \to T(0, 6)$ map $T(0, 6)$ onto the six sheets of the branch locus of the map from $V(2, 0)$ onto $V(0, 6)$. The modular groups $\Gamma(1, 1)$ and $\Gamma(1, 2)$ also have nontrivial centers which act trivially on $T(1, 1)$ and $T(1, 2)$, but which act on $V(1, 1)$ and $V(1, 2)$ by an involution on each fibre.

4. **Sections of** $\pi: V(p, n) \to T(p, n)$. John Hubbard has proved [**5**] that the map $\pi: V(p, 0) \to T(p, 0)$ has no holomorphic sections if $p \geqq 3$ and exactly six sections if $p = 2$. For $n \geqq 1$, the map

$$\pi: V(p, n) \to T(p, n)$$

has n holomorphic sections given, and it makes sense to ask whether π has a holomorphic section disjoint from the given ones (i.e., taking its values in $V(p, n)'$). We conjecture that no such sections exist unless $p = 1$ and $n = 1$ or 2. For the case $p = n = 1$, we can prove that only the obvious sections exist. We formulate that fact as

THEOREM 3. *Let* $U = \{z \in \mathbf{C}; \mathrm{Im}(z) > 0\}$ *be the upper halfplane. Suppose* $f: U \to \mathbf{C}$ *is a holomorphic function such that*

$$f(z) \neq m + nz \quad \textit{for all } z \in U, \textit{ all } m, n \in Z.$$

Then $f(z) = a + bz$, *where* a *and* b *are real and not both integers.*

Our proof of Theorem 3 follows the method of Hubbard in [**5**]. It would be interesting to have a direct proof.

ADDED IN PROOF. After submitting this paper, the author learned that M. Engber proved a stronger form of Theorem 1 independently in his 1972 Columbia thesis.

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