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PALEY-WIENER THEOREMS AND SURJECTIVITY OF INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES AND LIE GROUPS

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1. **Introduction.** The principal result of this paper is that if D is an invariant differential operator on a symmetric space X of the noncompact type then, for each function $f \in C^\infty(X)$, the differential equation $Du = f$ has a solution $u \in C^\infty(X)$. This is proved by means of a Paley-Wiener type theorem for the Radon transform on X. As a consequence we also obtain a Paley-Wiener theorem for the Fourier transform on X, that is an intrinsic characterization of the Fourier transforms of the functions in $C_c^\infty(X)$. In [**2**], Eguchi and Okamoto characterized the Fourier transforms of the Schwartz space on X. Invoking in addition the division theorem of Hörmander [**16**] and Lojasiewicz [**18**] we obtain by the method of [**11**] the surjectivity of D on the space of tempered distributions on X.

Finally, as a consequence of a structure theorem of Harish-Chandra [**8**] for the bi-invariant differential operators on a noncompact semisimple Lie group G, we obtain a local solvability theorem for each such operator.

2. **The range of invariant differential operators.** Let X be a symmetric space of the noncompact type, that is a coset space G/K where G is a connected, noncompact semisimple Lie group with finite center and K a maximal compact subgroup. Let $\boldsymbol{D}(X)$ denote the set of differential operators on X, invariant under G and let $C^\infty(X)$ denote the set of all C^∞ functions on X and $C_c^\infty(X)$ the set of $f \in C^\infty(X)$ of compact support.

THEOREM 2.1. *Let $D \neq 0$ in $\boldsymbol{D}(X)$. Then*

$$DC^\infty(X) = C^\infty(X).$$

As in Malgrange's proof of an analogous theorem for constant coefficient operators on $\boldsymbol{R}^n$ ([**3**], [**20**]) our proof proceeds by proving that if V is a closed ball in X then

$$f \in C_c^\infty(X), \operatorname{supp}(Df) \subset V \quad \textit{implies} \quad \operatorname{supp}(f) \subset V,$$

supp denoting support. This is proved by means of Theorem 2.2 below

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for the Radon transform [**10**] $f \to \hat{f}$ on X. If ξ is a horocycle in X then $\hat{f}(\xi)$ is the integral of f over ξ. The following Paley-Wiener type theorem for the Radon transform is the analog for X of Theorem 2.1 in [**12**]. The proof is however quite different and is in part based on Harish-Chandra's expansion for general Eisenstein integrals on G [**7**]. I am indebted to Harish-Chandra for communicating to me this expansion which has not been published, but will appear in [**22**]. It is a generalization of the asymptotic expansion for the spherical functions in [**6**].

THEOREM 2.2. *Let $L \in C_c^\infty(X)$ and let V be a closed ball in X. Assume $\hat{f}(\xi) = 0$ whenever the horocycle ξ in X is disjoint from V. Then $f(x) = 0$ for $x \notin V$.*

REMARK. Instead of assuming $f \in C_c^\infty(X)$ it suffices to assume that the function $g \to f(gK)$ belongs to the Schwartz space on G in the sense of [**9**, p. 19].

The analog of Theorem 2.1 for left invariant differential operators D on a Lie group L is in general false. In fact, it was proved to me by Hörmander in 1964 (independently proved in Cerèzo-Rouvière [**1**]) that if for a given L one assumes local solvability for every D then either L is abelian or has an abelian normal subgroup of codimension 1. However for each bi-invariant (i.e., left and right invariant) operator on the semisimple group G we have the following local solvability result.

THEOREM 2.3. *There exists an open neighborhood V of e in G with the following property: For each bi-invariant differential operator $D \neq 0$ on G,*

$$DC^\infty(V) \supset C_c^\infty(V).$$

The proof is easily deduced from a structure theorem for D (Harish-Chandra [**8** p. 477]) combined with Proposition 1.4 in Raïs [**21**] which deals with nilpotent groups.

3. **The Fourier transform on** X. Let $G = KAN$ be an Iwasawa decomposition of G, A and N being abelian and nilpotent, respectively. Let $\mathfrak{a}$ denote the Lie algebra of A, $\mathfrak{a}^*$ its dual and $\mathfrak{a}_c^*$ the complexification of $\mathfrak{a}^*$. If $\lambda \in \mathfrak{a}_c^*$ let $\operatorname{Im} \lambda$ denote its imaginary part. Let $|\lambda|$ denote the norm on $\mathfrak{a}^*$ given by the Killing form of the Lie algebra of G. If $H \in \mathfrak{a}$ the map $X \to [H, X]$ is an endomorphism of the Lie algebra $\mathfrak{n}$ of N whose trace we denote $2\rho(H)$. Let M be the centralizer of A in K, put $B = K/M$ and let db be the K-invariant measure on B with total measure 1. For $x \in X$, $b = kM \in B$, let $A(x, b) \in \mathfrak{a}$ be determined by $n \in N$, $x = kn \exp A(x, b)K$. Fixing a G-invariant measure dx on X the Fourier transform $\tilde{f}$ of a function f on X is defined by

$$\tilde{f}(\lambda, b) = \int_X e^{(-i\lambda+\rho)(A(x,b))} f(x)\,dx$$

for all $\lambda \in \mathfrak{a}_c^*$, $b \in B$, for which this integral converges absolutely [**13**]. It satisfies

$$(1) \qquad \int_B e^{(is\lambda+\rho)(A(x,b))} \tilde{f}(s\lambda, b)\,db \equiv \int_B e^{(i\lambda+\rho)(A(x,b))} \tilde{f}(\lambda, b)\,db$$

for $f \in C_c^\infty(X)$, and every element s in the Weyl group W of X, and the mapping $f \to \tilde{f}$ extends to an isometry of $L^2(X, dx)$ onto

$$L^2(\mathfrak{a}_+^* \times B, |c(\lambda)|^{-2}\,d\lambda\,db)$$

[**15**, pp. 120, 124]. Here $\mathfrak{a}_+^*$ is the positive Weyl chamber in $\mathfrak{a}^*$, $c(\lambda)$ is Harish-Chandra's c-function and $d\lambda$ is a suitably normalized Euclidean measure on $\mathfrak{a}^*$. Combining this characterization of $L^2(X)^\sim$ with Theorem 2.2, we obtain a characterization of the Fourier transforms of $C_c^\infty(X)$.

Definition. A C^∞ function $\psi(\lambda, b)$ on $\mathfrak{a}_c^* \times B$, holomorphic in λ, will be called a *holomorphic function of uniform exponential type* if there exists a constant $A \geqq 0$ such that, for each polynomial $P(\lambda)$ on $\mathfrak{a}_c^*$,

$$\sup_{\lambda \in \mathfrak{a}_c^*, b \in B} e^{-A|\mathrm{Im}\lambda|}\,|P(\lambda)\,\psi(\lambda, b)| < \infty.$$

Theorem 3.1. *The mapping $f \to \tilde{f}$ is a bijection of $C_c^\infty(X)$ onto the space of holomorphic functions of uniform exponential type satisfying* (1).

For the case when f is assumed K-invariant this reduces to a known result ([**4**, p. 434], for $\mathbf{SL}(2, \mathbf{R})$, [**14**], [**5**], [**15**, p. 37]). Finally, let $\mathscr{S}'(X)$ denote the dual space of the Schwartz space $\mathscr{S}(X)$. Its elements are distributions on X, the tempered distributions. In the manner indicated in the introduction we obtain an extension of Theorem 4.2 in [**11**].

Theorem 3.2. *Let $D \neq 0$ in $\mathbf{D}(X)$. Then*

$$D\mathscr{S}'(X) = \mathscr{S}'(X).$$

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