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CURVATURE AND COMPLEX ANALYSIS. II[1]

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This announcement is a continuation of Greene-Wu [**1**]; we shall present additional theorems relating curvature to function theory on noncompact Kähler manifolds. The first theorem improves Theorem 3 of [**1**].

THEOREM 1. *Let M be a complete simply connected Kähler manifold with nonpositive sectional curvature such that, for some $0 \in M$,*

$$|\textit{sectional curvature } (p)| \leqq C(d(0, p))^{-2-\varepsilon}$$

for some positive constants C and ε, where d is the distance function associated with the Kähler metric; then M admits no bounded holomorphic functions.

This theorem is false if $\varepsilon \leqq 0$. Indeed, on the unit disc, the Kähler metric $(1 - z\bar{z})^{-n} dz d\bar{z}$ (where n is any integer $\geqq 3$) is complete and its curvature function K satisfies $K < 0$ and $|K(z)| \leqq C(d(0, z))^{-2}$. ($0$ = origin of C.)

The next theorem and its corollary provide information about the absence of holomorphic p-forms ($p \geqq 1$) when the manifold is positively curved. For compact M, the result was known (Kobayashi-Wu [**6**]).

THEOREM 2. *Let M be a complete Kähler manifold of positive scalar curvature; then M possesses no holomorphic n-form in L^2 ($n = \dim M$). If the eigenvalues $r_1, \ldots, r_n$ of the Ricci tensor satisfy*

$$r_{i_1} + \ldots + r_{i_p} > 0 \quad \textit{for all } i_1 < \ldots < i_p,$$

then M admits no holomorphic p-form in L^2.

COROLLARY. (A) *If M is a complete Kähler manifold with positive Ricci*

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curvature, then M admits no holomorphic p-form in L^2 $(1 \leqq p \leqq \dim M)$. (B) *If M is a domain in* C^n *which admits a complete Kähler metric of positive scalar curvature, then M must have infinite Lebesgue measure.*

The next two theorems are concerned with the existence of holomorphic functions.

THEOREM 3. *Let M be a complete Kähler manifold with positive Ricci curvature and nonnegative sectional curvature. Furthermore, let L be a holomorphic line bundle on M with nonnegative curvature. Then* $H^p(M, \mathcal{O}(L)) = 0$ *for* $p \geqq 1$.

COROLLARY. (A) *Let M be a domain in* C^n *which admits a complete Kähler metric of positive Ricci curvature and nonnegative sectional curvature; then M is a Stein manifold.* (B) *Let M be a complete noncompact Kähler manifold with positive sectional curvature; then all the first and second Cousin problems on M are solvable.*

Some comments on this theorem follow. First, if M is compact, then this is a special case of Kodaira's vanishing theorem. Second, if the sectional curvature of M is actually positive, then one can even show $H^p(M, \mathcal{O}(T^{(\mu)} \otimes L)) = 0$ where L is as above and $T^{(\mu)}$ denotes the μth symmetric power of the holomorphic tangent bundle of M ($\mu \geqq 0$). When M is compact, this statement is a special case of a general theorem due to Griffiths [3, p. 212, Theorem G′]. Third, we conjecture that a noncompact complete Kähler manifold M of positive curvature is a Stein manifold.[2] We have proven this fact if M in addition possesses a *pole*, i.e. an $m \in M$ such that $\exp_m : M^m \to M$ is a diffeomorphism. Fourth, the proof of Theorem 3 hinges on a technical lemma which has the following easily stated consequence: *every convex function on a Kähler manifold is plurisubharmonic.* (A function on a Riemannian manifold is *convex* if and only if its restriction to each geodesic is a convex function of one variable.) This fact, which is so easy to prove when the Kähler manifold is C^n, turns out to be surprisingly subtle in the general case (see the forthcoming paper of Greene-Wu [2]).

For the statement of the next result, we need some notation. Let $A(M)$ be the algebra of holomorphic functions on M, let Ω be the volume form of M and let ρ be the distance (relative to the Kähler metric) from a fixed point $0 \in M$.

THEOREM 4. *Let M be an n-dimensional complete simply connected Kähler manifold whose sectional curvature is bounded between* $-d^2$ *and* 0. *Then for any* C^2 *plurisubharmonic function* φ *on M, the set*

[2] We have now proven this conjecture.

$$\left\{u \in A(M) : \int_M |u|^2(1+\rho^2)^N \exp\{-(2n-1)d^2\rho^2 - \varphi\}\Omega \right.$$

$$\left. < \infty \text{ for some integer } N\right\}$$

is dense in $A(M)$. *If* M *satisfies also* $-d^2 \leqq$ *sectional curvature* $\leqq -c^2 < 0$, *then the set*

$$\left\{u \in A(M) : \int_M |u|^2(1+\rho^2)^{-1} \exp\{-(2n-1)d^2\rho^2 - \varphi\}\Omega < \infty\right\}$$

is already dense in $A(M)$.

The first half of this theorem was essentially known to P. A. Griffiths (private communication); in the case $d = 0$ and hence $M = C^n$, the theorem is due to Hörmander [4, p. 119].

The next theorem is concerned with boundedness properties of the solutions of $\bar\partial u = f$. Again, the theorem is due to Hörmander if $d = 0$ ([4, p. 107], [5, p. 92]). Let us first explain the notation to come. For a continuous function φ on M and for an open subset D of M, we define

$$L^2_{(p,q)}(D, \varphi) = \left\{ f : f \text{ is a measurable } (p,q) \text{ form on } M \text{ such that } \int_D |f|^2 e^{-\varphi}\Omega < \infty \right\}.$$

$$L^2_{(p,q)}(D, \text{loc}) = \left\{ f : f \text{ is a measurable } (p,q) \text{ form on } M \text{ such that } \int_C |f|^2\Omega < \infty \text{ for any compact } C \subseteq D \right\}.$$

THEOREM 5. (A) *Let* M *be a simply connected complete Kähler manifold of dimension* n *such that* $-d^2 \leqq$ *sectional curvature* $\leqq 0$. *Let* D *be a bounded pseudoconvex open set in* M, *let* δ *be the diameter of* D *relative to the Kähler metric and let* φ *be a plurisubharmonic function in* D. *For every* $f \in L^2_{(0,q)}(D, \varphi)$, $q > 0$, *with* $\bar\partial f = 0$, *one can then find* $u \in L^2_{(0,q-1)}(D, \varphi)$ *such that* $\bar\partial u = f$ *and*

$$q \int_D |u|^2 e^{-\varphi}\Omega \leqq \left(\tfrac{1}{2}\delta^2 \exp(1 + \tfrac{1}{2}(2n-1)d^2\delta^2)\right) \int_D |f|^2 e^{-\varphi}\Omega.$$

(B) *Let* M *be as in* (A). *Let* φ *be any* C^2 *plurisubharmonic function on* M

and for a positive integer q, let $\tilde{\varphi} = \varphi + (2n - 1)qd^2\rho^2$. *If* $g \in L^2_{(0,q)}(M, \tilde{\varphi})$ *such that* $\bar{\partial} g = 0$, *then there exists* $u \in L^2{}_{(0,q-1)}(M, \text{loc})$ *such that* $\bar{\partial} u = g$ *and*

$$2\int_M |u|^2 e^{-\tilde{\varphi}}(1 + \rho^2)^{-2}\Omega \leqq \int_M |g|^2 e^{-\tilde{\varphi}}\Omega.$$

Finally, we give an improved version of Theorem 4 of [1]. First, we give a *new* definition of pseudo-Hermitian metrics. On a complex manifold M, g is called a *pseudo-Hermitian metric* if and only if (1) g is a continuous Hermitian bilinear form on M, (2) g is a C^2 Hermitian metric outside a proper subvariety S. The emphasis here is that g is only required to be a continuous tensor on M and that in specific examples, g will definitely fail to be differentiable on the singularity set S. By the *Ricci curvature* or *holomorphic sectional curvature of* g, we mean that of g restricted to $M - S$.

THEOREM 6. *A pseudo-Hermitian metric with nonpositive Ricci curvature on* C^n *must satisfy*

$$\limsup_{|z|\to\infty} |z|^2(\textit{Ricci curvature}\,(z)) > -\infty.$$

(*Here*, $|z|^2 = \sum_{i=1}^n z_i\bar{z}_i$.)

COROLLARY. *For* $n > 1$, *every pseudo-Hermitian metric on* C^n *must satisfy*

$$\limsup_{|z|\to\infty} |z|^2(\textit{holomorphic sectional curvature}\,(z)) > -\infty.$$

It remains to point out that Theorem 6 is false for an exponent > 2. Indeed, the pseudo-Hermitian metric $(1 + |z|^\delta)dzd\bar{z}$ on C (where δ is any positive constant) satisfies

$$\lim_{|z|\to\infty} |z|^{2+3\delta}(\textit{curvature}\,(z)) = -\infty.$$

(We take this opportunity to rectify some errors in [1]. (A) Theorem 1(iii) should be amended to read: If sectional curvature $\leqq -c^2 < 0$, then $dd^c\rho^2 \geqq (4 + 2c\rho \coth c\rho)\omega$, $dd^c \log(1 + \rho^2) > 0$ and outside $\{m : \rho(m) < 1\}$,

$$dd^c \log(1 + \rho^2) \geqq 2\{\rho c \coth \rho c - 1\}/(1 + \rho^2)$$

where coth denotes the hyperbolic cotangent. (B) The conclusion of Theorem 2(ii) should be

$$\int_{S_r} |f|^p \omega_r \geqq D_f \exp\{(2n - 1)^{1/2} cr\}$$

for $r \geqq 1$ and for some D_f which is independent of r and is positive if $f(0) \neq 0$.)

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