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ESTIMATES FOR THE SZEGÖ AND POISSON KERNELS OF SUFFICIENTLY ROUNDED TUBE DOMAINS

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In this paper we obtain estimates for the decrease at infinity of the Szegö and Poisson kernels, $S_\Gamma(X, Y) = S_{\Gamma;X}(Y)$ and $P_\Gamma(X, Y) = P_{\Gamma;X}(Y) = |S_\Gamma(X, Y)|^2 \|S_{\Gamma;X}\|_2^{-2}$, associated with proper cones $\Gamma \subset \boldsymbol{R}^n$ which are sufficiently smooth and satisfy certain curvature conditions. These estimates verify, for these cases, the conjecture of Stein (see [2], [4]) that the Poisson integral of an L^1 function converges restrictedly almost everywhere to that function on the distinguished boundary of a tube domain (Corollary IA). These and other results about the Poisson kernel will be elaborated on in [1].

Let Γ be a proper cone of $\boldsymbol{R}^n$ (that is, a nonempty, open, convex cone whose closure contains no whole line), Γ^* its dual cone

$$\Gamma^* = \{Y \in \boldsymbol{R}^n : (X, Y) > 0\ \forall X \in \bar{\Gamma} - \{0\}\}, \tag{1}$$

which is also proper, and $\Omega = \Omega_\Gamma$ its tube domain

$$\Omega = \Gamma \times i\boldsymbol{R}^n = \{Z \in \boldsymbol{C}^n : \operatorname{Re}(Z) \in \Gamma\}. \tag{2}$$

For $X \in \Gamma$ define the nonempty compact section $C_{\Gamma^*;X} = C_X^*$ of $\bar{\Gamma}^*$ as follows:

$$C_X^* = \{Y \in \bar{\Gamma}^* : (X, Y) = 1\} \subset \{Y : (X, Y) = 1\} \approx \boldsymbol{R}^{n-1}; \tag{3}$$

and similarly for $C_{\Gamma;Y} = C_Y$, $Y \in \Gamma^*$.

We will say Γ is C^N, $N \geqq 0$, if ∂C_Y is C^N. Γ will be said to satisfy the "flat curvature condition" if for some proper circular cone Δ of $\boldsymbol{R}^n$ and every $P \in \partial\Gamma$ there is a rotation ρ_P of $\boldsymbol{R}^n$ such that $P \in \partial(\rho_P\Delta)$ and $\rho_P\Delta \subset \Gamma$. The dual condition, the "sharp curvature condition," is stated similarly but reverses the last inclusion. We exclude, in our theorems, the trivial cases $n = 1, 2$.

THEOREM I. *Suppose Γ is a proper cone of $\boldsymbol{R}^n$, where*
(a) *$n = 3$ and Γ satisfies the flat curvature condition, or*
(b) *$n \geqq 4$, Γ is $C^{[n/2]}$, and Γ satisfies the sharp curvature condition.*

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Then, if we define m_Γ on $\boldsymbol{R}^n$ by

$$(4)\qquad \begin{aligned} m_\Gamma(Y) &= 1 \quad \text{if } |Y| \leqq 1, \\ &= |Y|^{-n/2}\{\max[1, \operatorname{dist}(Y, \partial\Gamma \cup -\partial\Gamma)]\}^{-1/2} \quad \text{if } |Y| \geqq 1, \end{aligned}$$

there exists for every compact $S \subset \Gamma$ a constant $k = k_S < \infty$ such that

$$(5)\qquad |S_{\Gamma;X}| \leqq k m_\Gamma \quad \forall X \in S,$$

and hence $k_2 = k_2(S) < \infty$ such that

$$(6)\qquad P_{\Gamma;X} \leqq k_2 m_\Gamma^2 \quad \forall X \in S.$$

COROLLARY IA. *If Γ satisfies the conditions of Theorem* I *and $f \in L^1(\boldsymbol{R}^n)$, and we define the Poisson integral of f by*

$$(7)\qquad Pf(X + iY) = (P_{\Gamma;X} * f)(Y), \qquad X \in \Gamma,\ Y \in \boldsymbol{R}^n,$$

then for almost every $Y_0 \in \boldsymbol{R}^n$, $Pf(Z) \to f(Y_0)$ as Z converges to iY_0 restrictedly in Ω_Γ.

(We say $Z \to Z_0$ restrictedly in Ω if $\{Z\} \subset \Omega$, $Z_0 \in \bar{\Omega}$, $Z \to Z_0$, and for some $\delta > 0$, $\operatorname{dist}(Z, \partial\Omega) \geqq \delta|Z - Z_0|\ \forall Z$.)

With a little more smoothness we attain an estimate which is the best possible, even for a circular cone (see [4]):

THEOREM II. *Suppose $n \geqq 3$ and Γ is a C^n proper cone of $\boldsymbol{R}^n$ satisfying the sharp curvature condition. Then the conclusions of Theorem* I *hold with $m_\Gamma(Y)$ replaced by*

$$(8)\qquad \begin{aligned} \mu_\Gamma(Y) &= 1 \quad \text{if } |Y| \leqq 1; \\ &= |Y|^{-n/2}\{\max[1, \operatorname{dist}(Y, \partial\Gamma \cup -\partial\Gamma)]\}^{-n/2} \quad \text{if } |Y| \geqq 1. \end{aligned}$$

SKETCH OF PROOFS. Since every C^2 proper cone satisfies the flat curvature condition, the hypotheses of Theorem I imply Γ^* satisfies the sharp curvature condition and is $C^{[n/2]}$ if $n \geqq 4$, while those of Theorem II imply Γ^* satisfies the sharp curvature condition and is C^n. These imply similar conditions for the C_X^*, uniformly for $X \in S$.

The crucial step is the radial integration of the usual formula for the Szegö kernel, which (see [2]) gives, for $S_X = S_{\Gamma;X}$,

$$(9)\qquad S_X(Y) = \frac{(n-1)!}{(2\pi)^n|X|}\int_{C_X^*} (1 + i(Y, \sigma))^{-n}\, d\sigma,$$

where $d\sigma$ is induced Lebesgue measure on the affine hyperplane $\{\sigma : (X, \sigma) = 1\}$. If we set $K(X) = (n-1)!/(2\pi)^n|X|$ we get that, for $p \neq 0$,

$$(10)\qquad S_X(pY) = \frac{K(X)}{p^n}\int_{C_X^*} (1/p + i(Y, \sigma))^{-n}\, d\sigma.$$

In Theorems I and II, the estimate is immediate for $|Y| \leqq 1$; thus, in (10), we may take $|Y| = 1$ and $p \geqq 1$. Then a gross estimate, using only the sharp curvature condition for Γ^*, yields the estimate of Theorem II for $Y \in \bar{\Gamma} \cup -\bar{\Gamma}$. For the case $p \geqq 1$, $|Y| = 1$, $Y \notin \Gamma \cup -\Gamma$, we use Fubini's theorem on (10) to get

$$S_X(pY) = \frac{K_2}{p^n} \int_a^b (1/p + iy)^{-n} \theta(y)\, dy \tag{11}$$

where $K_2 = K_2(X, Y)$, $a = a(X, Y)$, and $b = b(X, Y)$ are continuous, $a < 0 < b$, $[a, b] = (Y, C_X^*)$, and $\theta(y) = \theta(X, Y, y) = \lambda^{n-2}(\{\sigma \in C_X^* : (Y, \sigma) = y\})$ is smooth on (a, b) to the same degree as Γ^*, continuously in X and Y. λ^{n-2} is induced Lebesgue measure.

The results of Theorem I and II, and intermediate estimates for intermediate degrees of smoothness, now follow straightforwardly (except for some complications for $|a|$ or $|b|$ small) if we approximate θ at $y = 0$ by a polynomial $\tilde{\theta}$, do a contour integration on the analytic integral with the contour passing below the origin, and estimate the remainder term crudely. The hypothesis "C^N" could, for greatest generality, be replaced by "uniformly Lipschitz $N - 1$ order derivatives", which explains the apparent anomaly of Theorem I(a).

Corollary IA follows in the usual manner, by showing that the maximal operator $M = M_{\{Z\}}$ sending f into $Mf(Y) = \sup_{\{Z\}}\{|Pf(Z + iY)|\}$ is weak-type $(1, 1)$ if $\exists \delta > 0$ such that $Z \in \{Z\} \Rightarrow \operatorname{dist}(Z, C^n - \Omega) > \delta|Z|$. It suffices to show $\tilde{M}$ is weak-type $(1, 1)$, where $\tilde{M}f(Y) = \sup_{\delta > 0}(m_{\Gamma;\delta}^2 * |f|)(Y)$ and $m_{\Gamma;\delta}^2(Y) = \delta^{-n} m_\Gamma^2(Y/\delta)$. This result follows, as in [2], by majorizing m_Γ^2 in the natural fashion by a sum of multiples of characteristic functions of rectangles centered at 0, and applying 2.3 of [5].

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