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ON THE SUMMATION OF CONJUGATE FOURIER INTEGRALS OF FUNCTIONS OF SEVERAL VARIABLES

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Let K denote a homogeneous Calderón-Zygmund singular integral kernel $K(x)$ which is bounded and has mean value 0 on the unit sphere $S^{n-1} = \{x : |x| = 1\}$ of R^n and let $\hat{K}$ denote its principal valued Fourier transform (see [2]). In this note some simple propositions are stated which generalize results of V. L. Shapiro and others and can be proved in an elementary fashion. Suppose the Fourier transform of any integrable function f is defined by $\hat{f}(x) = \int f(y)e^{-ixy}\,dy$. The truncated kernel K_ε is defined by $K_\varepsilon(x) = K(x)$ if $|x| \geqq \varepsilon$ or $K_\varepsilon(x) = 0$ otherwise. For any function g not denoted by K and $\varepsilon > 0$ let $g_\varepsilon(x) = \varepsilon^{-n}g(\varepsilon^{-1}x)$, $g^\varepsilon(x) = g(\varepsilon x)$.

PROPOSITION 1. *Suppose $K(x) = \Omega(x)|x|^{-n}$ where Ω is homogeneous of degree 0, has mean value 0 on S^{n-1} and its modulus of continuity defined by*

$$\omega(t) = \sup\{|\Omega(\xi + h) - \Omega(\xi)| : |\xi| = 1, |h| \leqq t\}$$

satisfies the Dini condition $\int_0^1 \omega(t)\,dt/t < \infty$.

Furthermore suppose the integrable function φ is such that

$$\psi_0(x) = \int_{1 \leqq |x-y| \leqq |x|/2} K(x-y)\varphi(y)\,dy, \tag{1}$$

$$\psi_1(x) = \int_{|y| \geqq |x|} \varphi(y)\,dy, \tag{2}$$

$$\psi_2(x) = \int_{|y| \leqq 1} |y|^{-n}\,|\varphi(x-y) - \varphi(x)|\,dy \tag{3}$$

all satisfy

$$\int_0^\infty \sup_{t \leqq |x| \leqq 2t} |\psi_j(x)|\, t^{n-1}\,dt < \infty. \tag{4}$$

Finally suppose $\hat{\varphi} \in L^1$. Then, for $f \in L^1$ and $\int \varphi(x)\,dx = m$,

$$\lim_{\varepsilon \to 0}\left[(2\pi)^{-n}\int \hat{f}(y)\hat{K}(y)e^{ixy}\hat{\varphi}(\varepsilon y)\,dy - mK_\varepsilon * f(x)\right] = 0 \tag{5}$$

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for every point x in the Lebesgue set of f.

Proposition 1 can be shown to imply the following:

COROLLARY. (5) *holds at every point x in the Lebesgue set of f provided K is as in Proposition* 1, *$\varphi, \hat{\varphi}$ are integrable, ψ_2 defined by* (3) *satisfies* (4) *and*

(a) $\int_0^\infty \sup_{t \leqq |x| \leqq 2t} |\varphi(x)| (1 + \log^+ t) t^{n-1}\, dt < \infty$ *or*

(b) *ψ_1 satisfies* (4) *and there is a decomposition of R^n into a family of disjoint measurable sets E_j of bounded diameters $R^n = \bigcup_{j=0}^\infty E_j$ such that*

$$\int_{E_j} |\varphi(x)|\, dx \leqq \psi_3(\delta_j) |E_j|, \tag{6}$$

$$\left| \int_{E_j} \varphi(x)\, dx \right| \leqq \psi_3(\delta_j)(1 + \log^+ \delta_j)^{-1} |E_j|, \tag{7}$$

where ψ_3 satisfies $\int_0^\infty \sup_{t \leqq s \leqq 2t} \psi_3(s) t^{n-1}\, dt < \infty$, δ_j is the distance of E_j from 0 *and $|E_j|$ denotes the Lebesgue measure of E_j.*

Proposition 1 can be proved by means of the following fairly obvious lemma.

LEMMA. *Suppose Φ is measurable and*

$$\int_0^\infty \sup_{t \leqq |x| \leqq 2t} |\Phi(x)| t^{n-1}\, dt < \infty, \tag{8}$$

then

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon * f(x) = f(x) \int \Phi(x)\, dx$$

for every point x in the Lebesgue set of f.

PROOF OF PROPOSITION 1. $\hat{\varphi} \in L^1$ implies

$$\int \hat{f}(y) \hat{K}(y) e^{ixy} \hat{\varphi}(\varepsilon y)\, dy = (2\pi)^n (\text{p.v.}\, K * \varphi)_\varepsilon * f(x)$$

so that (5) amounts to $\lim_{\varepsilon \to 0} \Phi_\varepsilon * f(x) = 0$ where $\Phi = \text{p.v.}\, K * \varphi - mK_1$. Let

$$\Phi_0 = \text{p.v.}(K - K_1) * \varphi, \qquad \Phi_1 = K_1 * \varphi - mK_1.$$

By the above lemma it suffices to show that (8) is satisfied by Φ_j and $\int \Phi_j(x)\, dx = 0$ for $j = 0,1$. For Φ_0 these assertions follow directly from (4) for $j = 2$, the fact that K has mean value 0 on S^{n-1} and dominated convergence.

As for Φ_1 observe that $\Phi_1 = \sum_{m=1}^4 \Phi_{1m}$ where

$$\Phi_{11}(x) = \int_{|y|<|x|/2} (K_1(x-y) - K_1(x))\varphi(y)\,dy, \qquad \Phi_{12}(x) = \psi_0(x),$$

$$\Phi_{13}(x) = \int_{|y|\geqq|x|/2} K_1(x-y)\varphi(y)\,dy, \qquad \Phi_{14}(x) = -K_1(x)\psi_1(x/2).$$

Hence for $|x| > 2$,

$$\sup_{t\leqq|x|\leqq 2t} |\Phi_{11}(x)| \leqq Ct^{-n}\int_{|y|<|x|/2} \omega(|y|/t)|\varphi(y)|\,dy,$$

$$\sup_{t\leqq|x|\leqq 2t} |\Phi_{13}(x)|$$

$$\leqq C\|\Omega\|_\infty\left(t^{-n}\int_{t/2\leqq|y|\leqq 2t} |\varphi(y)|\,dy + \int_{|y|\geqq 2t} |\varphi(y)|\,|y|^{-n}\,dy\right).$$

It follows that

$$\int \sup_{t\leqq|x|\leqq 2t} |\Phi_{1m}(x)|t^{n-1}\,dt \leqq C\|\Omega\|_\infty \int |\varphi(x)|\,dx, \qquad m = 1, 3.$$

Furthermore by (4) for $j = 0, 1$, $\int_0^\infty \sup_{t\leqq|x|\leqq 2t} |\Phi_{1m}(x)|t^{n-1}\,dt$ is finite for $m = 2, 4$.

To see that $\int\Phi_1(x)\,dx = 0$ notice that if λ is a continuously differentiable radial function of compact support which equals 1 in a neighborhood of 0 then, for $\Phi_1^{(\varepsilon)} = (K_1\lambda^\varepsilon) * \varphi - mK_1\lambda^\varepsilon$, $\sup_{\varepsilon<1}|\Phi_1^{(\varepsilon)}|$ is still integrable and $\int\Phi_1^{(\varepsilon)}(x)\,dx = 0$ for any $\varepsilon > 0$, hence by dominated convergence $\int\Phi_1(x)\,dx = 0$.

Part (a) of the Corollary, for instance, implies Abel and Bochner-Riesz summability (above the critical index $(n-1)/2$) of "conjugate" Fourier transforms $\hat{K}\hat{f}$ of integrable functions f at any point where the singular integral p.v. $K * f$ exists and which is in the Lebesgue set of f (see [3], [4], [5], [6]). Part (b) could be used to prove the same assertion for $\hat{\varphi}(x) = (1-|x|^2)_+^{(n-1)/2}[\log(e/(1-|x|^2))]^{-\alpha}$ where $\alpha > 1$, in which case the Corollary may no longer apply.

The following result generalizes a result of Wheeden (see [7]) about Bochner-Riesz summability of conjugate Fourier transforms at the critical index.

PROPOSITION 2. *Suppose K is as in Proposition 1, φ is locally integrable, $\hat{\varphi} \in L^1$ and there is a decomposition of R^n into a family of disjoint measurable* *sets of bounded diameters* $R^n = \bigcup_{j=0}^{\infty} E_j$ *such that*

$$\int_{E_j} |\varphi(x)|\,dx \leqq B(1+\delta_j)^{-n}|E_j|,$$

$$\left|\int_{E_j} \varphi(x)\,dx\right| \leqq B(1+\delta_j)^{-n}(1+\log^+\delta_j)^{-1}|E_j|,$$

where δ_j *is the distance of* E_j *from* 0,

$$\lim_{s\to\infty}\int_{|x|<s}\varphi(x)\,dx = m.$$

Finally suppose for $0 < \varepsilon \leqq 1$

$$|x|^n \int_{|y|<\varepsilon} |y|^{-n}|\varphi(x+y)-\varphi(x)|\,dy \leqq B(\varepsilon)$$

where $B(\varepsilon) \leqq B$ *and* $\lim_{\varepsilon\to 0} B(\varepsilon) = 0$.

If $f \in L^1$ *and* $\int |f(x+y)-f(x)|\,|y|^{-n}\,dy < \infty$ *then* (5) *is valid.*

Since p.v. $K * \varphi$ belongs to the space C_0 of continuous functions vanishing at ∞ and since L^1 is dense in the space $\mathscr{M}^1$ of (complex) Borel measures of finite total variation with respect to the weak topology of the pairing with C_0 the above results generalize to conjugate Fourier-Stieltjes transforms. Furthermore by Poisson's summation formula Proposition 1 and the Corollary extend to periodic functions.

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