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SURFACES WITH PARALLEL MEAN CURVATURE VECTOR

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Let M be a surface immersed in a Riemannian manifold R^m of dimension m. Let D denote the covariant differentiation of R^m and $\boldsymbol{n}$ be a normal vector field on M. If we denote by $D^*\boldsymbol{n}$ the normal component of $D\boldsymbol{n}$, then D^* defines a connection in the normal bundle. A normal vector field $\boldsymbol{n}$ is called parallel if $D^*\boldsymbol{n} = 0$.

Let $\boldsymbol{H}$ and $\boldsymbol{h}$ denote the mean curvature vector and the second fundamental form of M in E^m. It is easy to see that minimal surfaces of a euclidean m-space E^m and minimal surfaces of hyperspheres of E^m are surfaces of E^m with parallel mean curvature vector, i.e. $D^*\boldsymbol{H} = 0$. On the other hand, for any analytic function $\varphi \not\equiv 0$ of $z = u + iv$, defined in a neighborhood of the origin in the (u, v)-plane, and constants α, β with $\alpha > 0$, Hoffman [3], [4] proved that, up to euclidean motions and isothermal coordinate $E(u, v)$, locally there exists one and only one surface in E^4, denoted by $M(\varphi, \alpha, \beta)$, with parallel mean curvature vector $\boldsymbol{H}$ such that $\alpha = |\boldsymbol{H}|$, and $\varphi = \varphi_3$, $\beta\varphi = \varphi_4$ where φ_3 and φ_4 are given in the Lemma of [3]. These surfaces are easy to check that they are contained in either an affine 3-space or an ordinary 3-sphere of E^m and they are neither minimal surfaces in E^m nor minimal surfaces of hyperspheres of E^m. Hence, the following problems seem to be interesting.

Problem I. Let M be a surface immersed in a euclidean m-space E^m with parallel mean curvature vector. If M is neither a minimal surface of E^m nor a minimal surface of a hypersphere of E^m, is M contained either in an affine 3-space of E^m or in an ordinary 3-sphere of E^m?

Problem II. If the answer to Problem I is in the affirmative, is M given locally by one of the surfaces $M(\varphi, \alpha, \beta)$?

The main purpose of this paper is to announce the following results. The details will appear elsewhere.

THEOREM I. *The answer to Problem* I *is in the affirmative.*

THEOREM II. *The answer to Problem* II *is in the affirmative.*

From theorem I we have the following corollaries.

COROLLARY 1. *Let M be a surface immersed in an m-sphere S^m with*

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parallel mean curvature vector. If M is neither a minimal surface of S^m nor a minimal surface of a small $(m - 1)$-sphere of S^m, then M must be a surface in a (small or great) 3-sphere of S^m with constant mean curvature.

This corollary follows immediately from Theorem I by imbedding S^m as a hypersphere of E^{m+1}.

COROLLARY 2. *Let M be a compact surface in E^m with parallel mean curvature vector and vanishing Gauss curvature. Then M is a product surface of two plane circles.*

This corollary follows immediately from Theorem 1 of [2] and a result of Lawson [5].

COROLLARY 3. *Let M be a complete surface in E^m with parallel mean curvature vector. If the Gauss curvature does not change sign, then M is one of the following surfaces:*

(i) *a minimal surface of E^m,*
(ii) *a minimal surface of a hypersphere of E^m,*
(iii) *a product surface of two plane circles, or*
(iv) *a product surface of a straight line and a plane circle.*

This corollary follows immediately from Theorem 2 of [3] and Theorem I.

Theorem II follows from Theorem I and the construction of $M(\varphi, \alpha, \beta)$ and Theorem I is based on the following lemmas.

LEMMA 1. *Let M be a surface immersed in E^m with parallel mean curvature vector and let R^N be the curvature tensor of the normal bundle. If $H \neq 0$, then either M is a minimal surface of a hypersphere of E^m or M has vanishing normal curvature tensor, i.e. $R^N = 0$.*

LEMMA 2. *Let M be a surface in E^m with parallel mean curvature vector and vanishing normal curvature tensor. Then M is contained in an affine 4-space of E^m.*

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