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NONASSOCIATIVE ADDITION OF UNBOUNDED OPERATORS AND A PROBLEM OF BREZIS AND PAZY

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ABSTRACT. We give a negative solution to a problem raised by Brezis and Pazy in the theory of nonlinear semigroups by relating it to a nonassociative phenomenon in the theory of addition of unbounded operators.

In their study of nonlinear contraction semigroups, Brezis and Pazy [**1**, p. 260] state the following problem:

"Let A, B and $A + B$ be maximal monotone sets and $F(t), G(t) \in \mathrm{Cont}(C)$ such that

$$\lim_{t\to 0}(I + (\lambda/t)(I - F(t)))^{-1}x = (I + \lambda A)^{-1}x, \tag{1}$$

$$\lim_{t\to 0}(I + (\lambda/t)(I - G(t)))^{-1}x = (I + \lambda B)^{-1}x \tag{2}$$

for every $\lambda > 0, x \in C$. Does

$$\lim_{t\to 0}(I + (\lambda/t)(I - F(t)G(t)))^{-1}x = (I + \lambda(A + B))^{-1}x \tag{3}$$

hold for every $\lambda > 0, x \in C$?"

Here C is a closed convex set in a Hilbert space and $\mathrm{Cont}(C)$ is the set of nonexpansive mappings of C into itself. Monotone sets are related to the (possibly multi-valued) generators of nonlinear contraction semigroups; however, in this note we will work only with linear semigroups, so we omit the detailed definition.

The answer to the above question is *no*, even in the linear theory, and even if B is assumed to be 0. It is quite interesting that this negative result is due to the failure of the associative law in generalized addition of operators; for this see [**3**].

To make the connection with product formulas, we note that (1) is equivalent to

$$\lim_{n\to\infty} F(t/n)^n x = e^{tA}x \tag{4}$$

for all x, uniformly on compact t intervals. For present purposes we require this only for linear operators; e^{tA} is the (C_0) contraction semigroup generated by A. This result is discussed in [**2**] and [**4**, Theorem IX, 3.6]; a nonlinear version is [**1**, Theorem 3.4]. In [**2**] it is shown that (4) holds if A is the closure of the strong derivative $F'(0)$, generalizing the original theorem of Trotter [**5**].

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We are now ready for the counterexample. Pick any skew-symmetric operator S on Hilbert space which admits two distinct skew-adjoint extensions A and A'; such operators are well known to exist in profusion. Take $F(t) = e^{tA}$, the unitary group generated by A. Take $G(t) = e^{-tA}e^{tA'}$. Note that the closure of $A' - A$ is 0. Hence by Trotter's theorem

$$G(t/n)^n \to I \tag{5}$$

in the strong operator topology, uniformly on compact t intervals; thus (2) holds with $B = 0$. It is quite trivial that

$$F(t/n)^n = (e^{t/nA})^n \to e^{tA}, \tag{7}$$

$$(F(t/n)G(t/n))^n = (e^{t/nA}e^{-t/nA}e^{t/nA'})^n \to e^{tA'}. \tag{8}$$

In other words (1) holds but (3) fails. In fact the analogue of (3) holds with A' replacing $A + B = A$.

As we pointed out in [**3**] this example illustrates the failure of the associative law for addition followed by closing:

$$A = [A + (-A + A')^-] \neq [(A - A)^- + A'] = A'.$$

References

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