BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, July 1972

BOOK REVIEWS

Hyperbolic Manifolds and Holomorphic Mappings by Shoshichi Kobayashi. 148 pp. Marcel Dekker, 1970. $11.75.

The theory of analytic functions of a complex variable is one of the most beautiful and pervasive subjects in mathematics. It has a strongly geometric aspect, which may roughly be labelled as the study of holomorphic mappings. One of the most important tools for this study is the Schwarz-Pick-Ahlfors lemma concerning the influence of curvature on such holomorphic mappings. The theme of this monograph is that the aforementioned lemma, together with elementary reasoning of a point-set topological character, leads to an interesting unified study of some properties of holomorphic mappings. The book is extremely well written, essentially selfcontained, and is recommended to beginning graduate students in mathematics as well as to specialists in complex variables. The reviewer's main criticism is that the deeper aspects of the use of differential-geometric methods for the study of holomorphic mappings, such as the Nevanlinna theory or the applications to algebraic geometry, go essentially unmentioned, so that a nonknowledgeable reader is left unaware of the enormous further possibilities of the field. I shall briefly discuss one part of the philosophy behind the methods presented in this book, shall then summarize some of the main results, and shall finally offer a few comments.

(1) A fundamental property of holomorphic functions is the Schwarz-Pick lemma, which states that a holomorphic mapping $w = f(z)$ from the unit disc $D = \{z : |z| < 1\}$ in the complex plane to itself is *distance decreasing* for the Poincaré metric $ds_D^2 = dz\, d\bar{z}/(1 - |z|^2)^2$. This result has been considerably generalized. To begin with, Ahlfors gave the following theorem: Let S be a Riemann surface having a conformal metric ds_S^2 whose Gaussian curvature K_S is everywhere $\leqq -4$ (recall that the Poincaré metric has constant negative curvature $K_D = -4$). Then a holomorphic mapping $f: D \to S$ is distance decreasing relative to the metrics ds_D^2 and ds_S^2. Because of the elementary fact that holomorphic sectional curvatures decrease on complex submanifolds, the Ahlfors theorem remains valid for a holomorphic mapping $f: D \to M$ where M is a complex manifold having an Hermitian metric ds_M^2 whose holomorphic sectional curvatures are everywhere negative and bounded away from zero. Finally the same distance decreasing result holds when D is an arbitrary bounded symmetric domain in C^n and ds_D^2 is the (suitably normalized) canonical invariant metric.

Continuing in a similar vein, it was proved by Chern and Kobayashi that, in the equidimensional case where $\dim_C D = \dim_C M$, a holomorphic mapping $f: D \to M$ is *volume decreasing* whenever D is a bounded symmetric domain and M satisfies a suitable curvature condition. This result, together with the distance decreasing statements, were finally unified and generalized by Lu who proved that all of the elementary symmetric functions of $f^*(ds_M^2)$ with respect to ds_D^2 are uniformly bounded when D is an Hermitian symmetric domain and M satisfies suitable negative curvature conditions.

These developments, which underlie much of the material in this book, fall under the general *principle of hyperbolic complex analysis*, as formulated by Chern:

> Holomorphic mappings into complex manifolds
> possessing suitable (negative) curvature properties
> satisfy very strong analytical conditions.

For example, using the modular function, the Schwarz-Pick lemma easily converts into the Schottky-Landau theorem: If $f(z)$ is a holomorphic function defined on the disc $|z| < R$, and if $f(z) \neq 0, 1$ and $|f'(0)| = 1$, then the radius R is subject to an inequality $R \leqq R_0$ where R_0 is a finite universal constant. Similarly, it is to be expected that the various generalizations of the Schwarz-Pick lemma discussed above, when interpreted in the framework of the principle of hyperbolic complex analysis, should lead to many interesting applications in the general study of holomorphic mappings. This monograph presents some of these applications, with special emphasis on those which may be deduced by elementary metric space methods from the various generalizations of the Schwarz-Pick lemma mentioned above.

(2) Turning now to some of the specific contents in the monograph, the Poincaré metric on the unit disc D may be used to give an intrinsic pseudodistance δ_M, the *Kobayashi metric*, on any complex manifold M. To define δ_M, we let p, q be two points on M, and we shall call a *chain* a sequence of holomorphic mappings $f_j: D \to M$ $(j = 1, \ldots, N)$ together with points $x_j, y_j \in D$ such that

$$f(x_1) = p, \qquad f_j(y_j) = f_{j+1}(x_{j+1}), \qquad f_N(y_N) = q,$$

and then

$$\delta_M(p, q) = \inf\left\{ \sum_{j-1}^{N} \rho(x_j, y_j) \right\},$$

where the "inf"is taken over all chains and ρ is the Poincaré distance on D. It is clear that a holomorphic mapping $f: M \to N$ is distance decreasing with respect to the Kobayashi metrics. We shall say that M is *hyperbolic* if δ_M is a true distance, and then the Ahlfors lemma states that M is hyperbolic if it carries a ds_M^2 whose holomorphic sectional curvatures are everywhere $\leqq -A$ for some positive constant A.

The notion of a hyperbolic complex manifold is a good one. The definition is simple, it has nice "functorial" properties, and it unifies and simplifies quite a few existing diverse results. For example, it follows immediately from the definitions that the polycylinder and unit ball in $\boldsymbol{C}^n$ are *not* biholomorphically equivalent for $n > 1$. Another beautiful recent result, due to Royden, is that the Kobayashi metric agrees with the Teichmuller metric on Teichmuller spaces, from which one derives several deep properties of these latter spaces. Finally, the procedure for defining the Kobayashi metric leads to intrinsic measures on the various subsets of any dimension in a complex manifold. These have been introduced and studied by Eisenman.

In this book, Kobayashi uses the hyperbolic manifolds to study automorphisms, minimal models, and removable singularities theorems for holomorphic mappings. To me, the latter are especially interesting, and are in response to the following question: Let $D^* = D - \{0\}$ be the punctured disc, M and N be complex manifolds with M a relatively compact open subset of N, and $f: D^* \to M$ a holomorphic mapping. Then when does f extend to a holomorphic mapping $f: D \to N$? The most classical case is when $N = \boldsymbol{P}_1$ is the Riemann sphere and

$$M = \boldsymbol{P}_1 - \{0, 1, \infty\}.$$

Then extending f is equivalent to the big. Pickard theorem. Examples due to Kiernan show that f does not always extend. The basic result here, due to Kwack, Kobayashi, and Ochiai, goes as follows: The mapping $f: D^* \to M$ extends whenever (i) M is hyperbolic, and (ii) if $p, q \in \partial M$ and if $\{x_n\}, \{y_n\} \subset M$ are sequences with $x_n \to p$, $y_n \to q$, and $\delta_M(X_n, y_n) \to 0$, then $p = q$. If $M = N$ is compact, then (ii) follows from (i) and we obtain the theorem of Mrs. Kwack. Taking $M = \boldsymbol{P}_1 - \{0, 1, \infty\}$ and $N = \boldsymbol{P}_1$ we have the big Picard theorem.

(3) My comments are just a few minor corrections and clarifications.

(a) Proposition 1.1 on p. 106 is false as stated (there are nontrivial holomorphic line bundles on $\boldsymbol{C}^2 - \{0\}$). This proposition is used on p. 107, and the desired application is true, but the proof seems to require something like the Remmert-Stein theorem. Aside from this, the book is remarkably free of errors and misprints.

(b) The terminology "Poincaré-Bergman metric" is introduced for the non-Euclidean metric $dz\, d\bar{z}/(1 - |z|^2)^2$ on the disc $|z| < 1$. This metric was used systematically by Poincaré, and then by many others, from 1881 onward, while Bergman's kernel function was discovered c. 1920. It seems to me that the more standard terminology "Poincaré metric" would have been more suitable.

(c) Problem 1 on p. 131 has been solved by Mark Green, who has obtained best possible degeneracy results for holomorphic maps of C^k into P_n which omit a certain number of hyperplanes in general position.

(d) The proof of Theorem 3.1 on p. 83 is due to Grauert and Reckziegel, and it was Mrs. Kwack who recognized that their argument had wider implications than those which they gave.

(e) Finally, I should like to offer a clarification concerning the result mentioned in example 2 on p. 94. Let D be a bounded symmetric domain in C^n, Γ an arithmetic group of automorphisms of D, and $M = D/\Gamma$ the quotient. There are two compactifications N_1 and N_2 of M due respectively to Baily-Borel-Satake and Pyatetzki-Shapiro. For the first, there is a fairly complicated description of the fundamental domain Ω_1 for Γ acting on D, and, using this, N_1 turns out to be a Hausdorff space which carries the structure of a complex-analytic variety. For the second, there is a much easier description of the fundamental domain Ω_2. It has just been recently proved by A. Borel that the natural mapping $h: N_1 \to N_2$ is a homeomorphism, so that the compactifications coincide. The extension theorem of Borel is for $f: D^* \to N_1$, and the proof is difficult because of the complicated nature of Ω_1. The Kobayashi-Ochiai result is for $f: D^* \to N_2$, and yields Borel's theorem only by using the identification $N_1 \tilde{\to} N_2$. The extension theorem of Borel is of great use in algebraic geometry, and has recently been used by P. Deligne to prove the Riemann hypothesis for algebraic K3 surfaces.

(f) Kiernan and Kobayashi have recently proved that a holomorphic mapping $f: D/\Gamma \to D'/\Gamma'$ between arithmetic quotients of bounded domains extends to a holomorphic mapping $f: N_2 \to N_2'$ between the compactifications. This result will appear in Ann. of Math.

(g) The answer to problem 2 appears in the papers by H. Cartan (Ann. École Norm Sup. **45** (1928), 256-346) and A. Bloch (Ann. École Norm Sup. **43** (1926), 309-362).

(h) The result in Kobayashi-Ochiai [2] originally appeared in P. Griffiths (Ann. of Math. **93** (1971), 439-458).

P. GRIFFITHS

Combinatorial identities by John Riordan. John Wiley and Sons, New York, 1968.

1. In the eighth book of his celebrated work, the geographer Strabo essays a detailed description of the whole of Greece. Strabo is well aware of the fact that his task is not completely straightforward, since it involves him in some highly conjectured identifications of sites mentioned by