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GENERALIZED RAMSEY THEORY FOR GRAPHS

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The classical Ramsey numbers [7] involve the occurrence of monochromatic complete subgraphs in line-colored complete graphs. By removing the completeness requirements and admitting arbitrary forbidden subgraphs within any given graph, the situation is richly and nontrivially generalized.

The Ramsey number $r(m, n)$ as traditionally studied in graph theory [5, p. 15] may be defined as the minimum number p such that every graph with p points which does not contain the complete graph K_m must have n independent points. Alternatively, it is the smallest p for which every coloring of the lines of K_p with two colors, green and red, contains either a green K_m or a red K_n. Thus the *diagonal Ramsey numbers* $r(n, n)$ can be described in terms of 2-coloring the lines of K_p and regarding K_n as a forbidden monochromatic subgraph without regard to color.

This viewpoint suggests the more general situation in which an arbitrary graph G has a c-coloring of its lines and the number of monochromatic occurrences of a forbidden subgraph F (or of a forbidden family of graphs) is calculated. A host of problem areas within graph theory can be subsumed under such a formulation. These include the line-chromatic number, in which the 3-point path is forbidden. The arboricity of G involves forbidding all cycles. The thickness of a graph forbids the Kuratowski graphs. Complete bipartite graphs can be taken for both G and F, and so can cubes Q_n and Q_m.

There has long been a sentiment in graph theory that there is an intimate relationship between extremal graph theory and Ramsey numbers. It does not appear possible to derive either Turán's theorem or Ramsey's theorem from the other. However, we show that extremal bipartite graph theory does in fact imply the bipartite form of Ramsey's theorem.

Let $\mathscr{F}$ be a family of graphs, G a given graph, and c a positive integer. We denote by $R(G, \mathscr{F}, c)$ the greatest integer n with the property that, in every coloration of the lines of G with c colors, there are at least n monochromatic occurrences of a member of $\mathscr{F}$. Without any loss of generality, we can assume that every forbidden subgraph $F \in \mathscr{F}$ has no isolated points. If $\mathscr{F}$ contains just one forbidden subgraph F then we write simply $R(G, F, c)$ instead of $R(G, \{F\}, c)$.

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Some results on R-numbers already exist. For example, Goodman [**4**] found the exact value of $R(K_n, K_3, 2)$, Erdös [**2**] obtained a probabilistic lower bound for $R(K_n, K_m, 2)$, and Moon and Moser [**6**] calculated a lower bound for $R(K_{n,n}, K_{2,2}, 2)$ which they conjectured to be exact.

Applying a proof technique of Erdös [**1**] to a more general situation with arbitrary $\mathscr{F}$ and c colors, we obtain the following implication in which $q(F)$ is the number of lines in graph F.

THEOREM 1. *$\sum_{F \in \mathscr{F}} R(G, F, 1)c^{-q(F)+1} < 1$ implies $R(G, \mathscr{F}, c) = 0$.*

The *cube Q_n* is defined as usual as the graph with 2^n points which can be taken as all binary n-sequences, with two points adjacent whenever their sequences differ in just one place.

THEOREM 2.

$$R(Q_n, Q_m, c) = \binom{n}{m} 2^{n-m}, \qquad \text{if } \min(m, c) = 1,$$
$$= 0, \qquad \text{otherwise.}$$

Another result, concerning R-numbers for trees, is as follows. For any two trees T_1 and T_2 such that T_1 is not a star,

$$R(T_2, T_1, c) = 0 \quad \text{whenever } c \geqq 2.$$

On the other hand, if $T_1 = K_{1,m}$ is a star and $d_1, d_2, \ldots, d_n$ is the degree sequence of T_2, then $R(T_2, T_1, c)$ depends only on the degree sequence of T_2 and is easily computed.

In general, for c families $\mathscr{F}_i$ of graphs with no isolated points, the number $r(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_c)$ is the minimum p such that in every c-coloring of the lines of K_p, there exist i and $F \in \mathscr{F}_i$ such that all the lines of F have color i. In particular, the number $r(F_1, F_2)$ is the minimum p such that every 2-coloring of K_p contains a green F_1 or a red F_2. With the help of a general

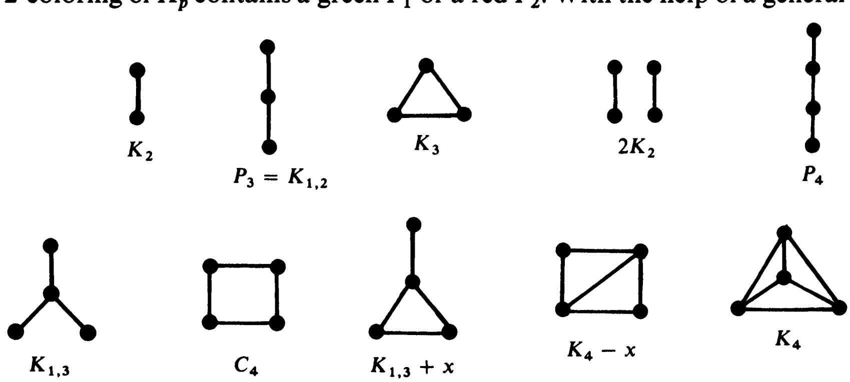

FIGURE 1

lower bound, the exact values of $r(F_1, F_2)$ are determined for all graphs F_i with less that five points having no isolates.

There are exactly ten graphs F (Fig. 1) with at most 4 points, having no isolates. For convenience in identifying them, we use the operations on graphs from [**5**, p. 21] to write a symbolic name for each.

THEOREM 3. *Let F_1 and F_2 be two graphs (not necessarily different) with no isolated points. Let k be the number of points in a largest connected component of F_1, and let χ be the chromatic number of F_2. Then the following lower bound holds*:

$$r(F_1, F_2) \geqq (k - 1)(\chi - 1) + 1.$$

The following table summarizes the results we have obtained for the 10 small graphs shown in Fig. 1.

Table 1. *Small generalized Ramsey numbers $r(F_1, F_2)$*

	K_2	P_3	$2K_2$	K_3	P_4	$K_{1,3}$	C_4	$K_{1,3}+x$	K_4-x	K_4
K_2	2	3	4	3	4	4	4	4	4	4
P_3		3	4	5	4	5	4	5	5	7
$2K_2$			5	5	5	5	5	5	5	6
K_3				6	7	7	7	7	7	9
P_4					5	5	5	7	7	10
$K_{1,3}$						6	6	7	7	10
C_4							6	7	7	10
$K_{1,3}+x$								7	7	10
K_4-x									10	11
K_4										18

The results stated above without proof will be demonstrated in a series of papers, along with other related theorems. We conclude with a few open problems:

1. Our experience seems to indicate that

$$r(F_1, F_2) \geqq \min(r(F_1, F_1), r(F_2, F_2)).$$

We conjecture that this always holds.

2. We note that the exact determination of the number $r(K_5, K_5) = r(5, 5)$ remains beyond the scope of present methods. It is our hope that our more general approach may provide a bootstrap method.

3. Very recently, Chartrand and Schuster [**8**] proposed the problem of determining the numbers $r(C_m, C_n)$ where C_p is the graph consisting of just one cycle with p points. They denoted this number by $c(m, n)$ and found

its values for $m = 3, 4, 5$ and also announced that $c(6, 6) = 8$. The remaining values of $c(m, n)$ are open. Their determination has independently attracted the interest of J. A. Bondy, W. G. Brown, and P. Erdös.

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