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DE RHAM'S INTEGRALS AND LEFSCHETZ FIXED POINT FORMULA FOR d'' COHOMOLOGY

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We give here a brief sketch of a different approach to the Atiyah-Bott type Lefschetz fixed point formula for Dolbeault complexes. Our method is based on an extension to the complex case of de Rham's integral formulas for Kronecker indices [7]. This approach yields results for general fixed point sets, and in particular we shall give here a formula for isolated degenerate fixed points. Details and related results will appear elsewhere.

Following notations in [**1**], [**2**], [**3**], let X be a compact complex analytic manifold of complex dimension n,

$$\Gamma(\Lambda^{p,*}X):0 \to \Gamma(\Lambda^{p,0}) \xrightarrow{d''} \Gamma(\Lambda^{p,1}) \to \cdots \xrightarrow{d''} \Gamma(\Lambda^{p,n}) \to 0,$$

$0 \leqq p \leqq n$, the pth Dolbeault complex, $f:X \to X$ a complex analytic mapping with isolated fixed points, and

$$T_{p,q} = \Lambda^p(d'f^*) \otimes \Lambda^q(d''f^*) \circ f^*:\Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p,q})$$

the induced endomorphisms on the complex. In terms of $T_{p,q}$ we define, as in [**3**],

$$\operatorname{graph}\{T_{p,q}\} \in \Gamma'(\Lambda^{p,q} \boxtimes (\Lambda^{p,q})')$$

where $(\Lambda^{p,*})'$ denotes the geometric dual and Γ' the space of distributions. It is then seen that

$$\operatorname{graph}\{T_p\} = \sum_{q=0}^{n} \operatorname{graph}\{T_{p,q}\} \in H'(\Lambda^{p,*} \boxtimes (\Lambda^{p,*})').$$

Similarly define

$$\Delta_p = \sum_{q=0}^{n} \operatorname{graph}\{I_{p,q}\} \in H'((\Lambda^{p,*})' \boxtimes \Lambda^{p,*})$$

where $I_{p,q}:\Gamma((\Lambda^{p,q})') \to \Gamma((\Lambda^{p,q})')$ is the identity. Analogous to [**3**], [**6**], one deduces from Poincaré duality and Künneth formula that the Lefschetz number

$$L(f^{p,*}) = \sum (-1)^q \operatorname{trace}\{T^*_{p,q}\}$$

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is given by

(1) $$L(f^{p,*}) = (\text{*graph}\,\{T_p\}, \bar{\Delta}_p)$$

where the inner product is defined as usual by $(\alpha, \beta) = \int \alpha \Lambda {*}\bar{\beta}$.

The product (*graph $\{T_p\}$, $\bar{\Delta}_p$) is determined at the intersection of singular supports of the two distributions, and may be computed locally. Let a local coordinate map be given, through which the fixed point is mapped to origin in C^{2n}, the piece of singular support of Δ_p is mapped to a subset V of the diagonal, and that of graph $\{T_p\}$ is mapped to a set U. Denote by graph $\{T_p\}_U$, $(\Delta_p)_V$ the distributions transformed to C^{2n}. Since in euclidean space $(d\delta + \delta d)G = 1$ [7], and $d\delta + \delta d = 2(d'\delta' + \delta' d') = 2(d''\delta'' + \delta'' d'')$ we can write

$$\begin{aligned}(\text{*graph}\,\{T_p\}_U, \bar{\Delta}_{pV}) &= 2(d'\delta'{*}G\,\text{graph}\,\{T_p\}_U, \bar{\Delta}_{pV})\\ &\quad + 2(\text{*graph}\,\{T_p\}_U, \delta' d' G\bar{\Delta}_{pV})\end{aligned}$$

and the r.h.s. is given by integrals of smooth functions. The sum is invariant as we increase the support of V to the full diagonal Δ in C^{2n}, and we find the second term vanishes while the first term becomes

(2) $$2\int_{\Delta\zeta}\int_{\partial U_z} \delta''_z {*}_\zeta P_\zeta g(z, \zeta)$$

where $g(z, \zeta)$ is the Green's form in C^{2n}, and

$$P:\Lambda \otimes \Lambda \to \sum_q \Lambda^{n-p,n-q} \otimes \Lambda^{p,q}$$

is the projection determined by Δ_p.

Suppose now the mapping is described locally by

$$z_{n+i} = f_i(z_1, \ldots, z_n), \qquad 1 \leqq i \leqq n,$$

and denote $h_i(z) = z_i - f_i$. Let $A_p(z)$ be holomorphic functions defined by

$$\sum A_p(z)t^{n-p} = \det\left(tI + \left(\frac{\partial f_i}{\partial z_j}\right)\right).$$

Then (2) is evaluated to be

(3) $$\frac{(n-1)!}{(2\pi i)^n}\int_{\partial U} A_p(z) \frac{\sum \bar{h}_j d\bar{h}_1 \wedge dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_j \wedge d\bar{h}_{j+1} \wedge \cdots \wedge d\bar{h}_n \wedge dz_n}{(\sum |h_i|^2)^n}.$$

In the case of a simple fixed point, a change of variable together with Bochner's integral formula [4] applied to (3) yields the formula (4.9) of [2].

In the case of an isolated nonsimple fixed point, we shall give in a subsequent paper an algorithm for computing (3). It will be seen that in this case, the algorithm gives the same computation as Grothendieck's residue symbol [**5**]. In the latter's notation (3) can be written as:

$$\operatorname{Res}\left[\frac{A_p(z)dz_1 \wedge \cdots \wedge dz_n}{h_1 \cdots h_n}\right].$$

A cruder and quite different approach to this problem is given in [**8**].

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