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PROOF OF EDREI'S SPREAD CONJECTURE

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A few years ago A. Edrei introduced the notion of the spread of a deficient value:

Let $f(z)$ be a meromorphic function of lower order $\mu < \infty$. (Recall that $\mu = \liminf_{r\to\infty} (\log T(r,f))/(\log r)$.) Let $\{r_m\}$ be a sequence of Pólya peaks of order μ for $f(z)$ (see [**1**] for the definition of Pólya peaks), and let τ be a deficient value for $f(z)$, $\delta(\tau, f) = \delta > 0$. Put

$$s_m(\infty, f) = \operatorname{meas}\{\theta \in [-\pi, \pi] : |f(r_m e^{i\theta})| > r_m\},$$

$$s_m(\tau, f) = s_m(\infty, (f-\tau)^{-1}) \qquad (\tau \neq \infty).$$

The *spread* of τ is defined as

$$\sigma(\tau, f) = \liminf_{m\to\infty} s_m(\tau, f).$$

Edrei conjectured [**2**, p. 57] the

Spread relation.

$$\sigma(\tau, f) \geqq \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\sqrt{\frac{\delta}{2}}\right\}.$$

He obtained an approximation [**1**, p. 83] of this inequality good enough to yield assertion I of Theorem 2 below.

The author has now obtained a proof of the exact form of the spread relation. A principal tool in this proof is the following theorem, which seems to be of independent interest.

THEOREM 1. *Let $f(z)$ ($\not\equiv 0$) be a meromorphic function. Put*

$$m^*(z) = \sup_E \frac{1}{2\pi}\int_E \log|f(re^{i\omega})|\,d\omega \qquad (z = re^{i\theta}, 0 < \theta < \pi),$$

where the sup *is taken over all sets E with measure exactly 2θ.*

Then the function

$$T^*(z) = m^*(z) + N(|z|, f)$$

is subharmonic in the upper half plane $\operatorname{Im} z > 0$.

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The establishment of the spread relation makes it possible for Edrei to give a solution of the "deficiency problem" for functions of small order.

THEOREM 2 (EDREI). *Let $f(z)$ be a meromorphic function of lower order μ, $0 < \mu \leqq 1$.*

I. *If $0 < \mu \leqq \frac{1}{2}$, then, either*

$$\sum_{\tau} \delta(\tau, f) \leqq 1 - \cos \pi\mu,$$

or else f has only one deficient value (with deficiency $> 1 - \cos \pi\mu$).

II. *If $\frac{1}{2} < \mu \leqq 1$, then*

$$\sum_{\tau} \delta(\tau, f) \leqq 2 - \sin \pi\mu.$$

Equality holds if and only if f has only two deficient values, one of deficiency 1, the other of deficiency $1 - \sin \pi\mu$.

REFERENCES

1. A. Edrei, *Sums of deficiencies of meromorphic functions*, J. Analyse Math. **14** (1965), 79–107. MR **31** #4909.

2. ———, *Sums of deficiencies of meromorphic functions*. II, J. Analyse Math. **19** (1967), 53–74. MR **35** #6831.

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