BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

RIEMANNIAN MANIFOLDS OF FINITE ORDER

BY RICHARD HOLZSAGER[1]

Communicated by Raoul Bott, October 5, 1971

ABSTRACT. Several types of Riemannian manifolds are characterized by the growth of area of displacements of hypersurfaces along normal geodesics.

If H is a compact hypersurface with oriented normal bundle in a Riemannian manifold M and H_s is the (possibly singular) hypersurface of points at distance s along normal geodesics, then let $A_H(s)$ be the area of H_s. In [4], [3], [2], the functions A_H were used to give characterizations respectively of the Euclidean plane, surfaces of constant curvature, manifolds of constant sectional curvature. A different proof, yielding further results, is outlined here.

Say that a manifold M has finite order r if there is a linear differential equation of order r with constant coefficients which is satisfied by A_H for every hypersurface H and r is the least such integer. If there is no such differential equation, say that M has infinite order.

THEOREM 1. (a) $\operatorname{ord} M \geqq \dim M$;

(b) $\operatorname{ord} M = \dim M \Leftrightarrow M$ *has constant sectional curvature*;

(c) $\operatorname{ord} M = 1 + \dim M \Leftrightarrow M$ *is locally isometric to a complex projective space other than* CP^1, *or to its dual symmetric space*;

(d) *if* $\dim M = 2$, $\operatorname{ord} M < \infty \Leftrightarrow M$ *has constant curvature*;

(e) *if* M *is symmetric*, $\operatorname{ord} M < \infty \Leftrightarrow M$ *has rank* 1 *or is flat.*

The first step of the proof is to choose a point x in H, take an orthonormal frame $E_1, \ldots, E_n$ at x with E_n normal to H (where $n = \dim M$) and parallel translate this frame along the normal geodesic through x. (A similar moving frame is also used in [1].) Let f_s be the obvious map $H \to H_s$ and $T_1, \ldots, T_{n-1}$ a moving frame along, and orthogonal to, the same geodesic, with $T_i(f_s(x)) = df_s(T_i(x))$. It can be shown that if we define functions t_{ij} $(1 \leqq i, j \leqq n-1)$ by $T_i = \sum t_{ij}E_j$, then $t''_{ij} = \sum t_{ik}c_{kj}$, where $c_{kj} = \langle R(E_n, E_k)E_n, E_j\rangle$. The "if" portions of (b), (c), (d), and (e) now follow quite directly.

LEMMA. *If in a symmetric space* M, *the eigenvalues of the bilinear form* $\langle R(E, -)E, -\rangle$ *are the same for all unit vectors* E, *then* M *has rank* 1 *or is flat.*

AMS 1970 *subject classifications.* Primary 53B20, 53B25; Secondary 53C35, 34A30.
Key words and phrases. Order of a manifold, constant curvature, rank 1, symmetric space.
[1] This work was supported by a grant from The American University.

The "only if" portion of (e) follows from the Lemma, and the rest of Theorem 1 follows from Theorem 2.

THEOREM 2. *Let C be a symmetric $(n-1) \times (n-1)$ matrix of C^∞ functions. Suppose that for every $(n-1) \times (n-1)$ solution T of the equation $T'' = TC$,* $\det T$ *is a solution of a given linear differential equation of order r with constant coefficients (and r minimal). Then*

(a) $r \geqq n$;

(b) $r = n \Leftrightarrow C$ *is a constant multiple of the identity matrix*;

(c) $r = 1 + n \Leftrightarrow C$ *is constant and is orthogonally similar to a diagonal matrix with diagonal entries $a, a, \ldots, a, 4a$*;

(d) $n = 2 \Rightarrow C$ *is constant.*

Beyond these results, I have the following conjectures:

CONJECTURE. The matrix C in Theorem 2 is constant for any r, n.

COROLLARY (TO CONJECTURE). *For any M,* $\operatorname{ord} M < \infty \Leftrightarrow M$ *is locally symmetric of rank* 1 *or is flat.*

In proving that the Corollary follows from the Conjecture, the following lemma is useful.

LEMMA. *Let M be a Riemannian manifold such that for any pair of vectors X, Y based at the same point, $\langle (D_X R)(X, Y)X, Y \rangle = 0$. Then M is locally symmetric.*

REFERENCES

1. N. Grossman, *On characterization of Riemannian manifolds by growth of tubular neighborhoods* (mimeographed preprint).
2. R. Holzsager, *A characterization of Riemannian manifolds of constant sectional curvature*, J. Differential Geometry (to appear).
3. R. Holzsager and H. Wu, *A characterization of two-dimensional Riemannian manifolds of constant curvature*, Michigan Math. J. **17** (1970), 297–299.
4. H. Wu, *A characteristic property of the euclidean plane*, Michigan Math. J. **16** (1969), 141–148. MR **39** #7544.

DEPARTMENT OF MATHEMATICS, THE AMERICAN UNIVERSITY, WASHINGTON, D.C. 20016