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RIGIDITY THEOREMS FOR SURFACES IN EUCLIDEAN SPACE

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Let M be a surface immersed in euclidean m-space E^m, and let ∇ and ∇' be the covariant differentiations of M and E^m respectively. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two tangent vector fields on M. Then the second fundamental form $\boldsymbol{h}$ is given by

$$\nabla'_{\boldsymbol{u}}\boldsymbol{v} = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{h}(\boldsymbol{u}, \boldsymbol{v}). \tag{1}$$

If $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_m$ is a local field of orthonormal frame such that $\boldsymbol{e}_1, \boldsymbol{e}_2$ are tangent to M and $\boldsymbol{e}_3, \ldots, \boldsymbol{e}_m$ are normal to M, then the mean curvature vector $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = \frac{1}{2}\sum_{i=1}^{2} \boldsymbol{h}(\boldsymbol{e}_i, \boldsymbol{e}_i). \tag{2}$$

For a normal vector field $\boldsymbol{\eta}$ and a tangent vector field $\boldsymbol{u}$ on M, let $\nabla^*_{\boldsymbol{u}}\boldsymbol{\eta}$ denote the normal component of $\nabla'_{\boldsymbol{u}}\boldsymbol{\eta}$. Then ∇^* defines a connection in the normal bundle of M in E^m. A normal vector field $\boldsymbol{\eta}$ is said to be parallel in the normal bundle if $\nabla^*\boldsymbol{\eta} = 0$. Let h^r_{ij}, $i, j = 1, 2, r = 3, \ldots, m$, be the coefficients of the second fundamental form $\boldsymbol{h}$. Then the Gauss curvature K and the normal curvature K_N are given by

$$K = \sum_{r=3}^{m} (h^r_{11}h^r_{22} - h^r_{12}h^r_{12}), \tag{3}$$

$$K_N = \sum_{r,s=3}^{m} \left[\sum_{k=1}^{2} (h^r_{1k}h^s_{2k} - h^r_{2k}h^s_{1k})\right]^2, \tag{4}$$

respectively. The mean curvature vector $\boldsymbol{H}$, the Gauss curvature K, and the normal curvature K_N play important roles, in differential geometry, for surfaces in euclidean space.

Let $\langle\, , \rangle$ denote the scalar product of E^m. If the mean curvature vector $\boldsymbol{H}$ is nowhere zero and there exists a function f on M such that $\langle \boldsymbol{h}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{H}\rangle = f\langle \boldsymbol{u}, \boldsymbol{v}\rangle$ for all tangent vector fields $\boldsymbol{u}, \boldsymbol{v}$ on M, then M is called a *pseudo-umbilical surface* of E^m.

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We shall now define the Veronese surface. Let (x, y, z) be the natural coordinate system in E^3 and $(u^1, u^2, u^3, u^4, u^5)$ the natural coordinate system in E^5. We consider a surface in E^5 given by

$$\left(\frac{c}{\sqrt{3}}xy, \frac{c}{\sqrt{3}}yz, \frac{c}{\sqrt{3}}xz, \frac{c}{2\sqrt{3}}(x^2 - y^2), \frac{c}{6}(x^2 + y^2 - 2z^2)\right),$$

$$x^2 + y^2 + z^2 = 3, \qquad c = \text{constant} > 0.$$

This surface is a real projective plane in E^5 with $\nabla^* \boldsymbol{H} = 0$, $K = \frac{1}{3}c^2$ and $K_N = \frac{8}{9}c^4$. It is called the *Veronese surface.*

The main purpose of this paper is to announce two rigidity theorems for surfaces in euclidean space with mean curvature vector parallel in the normal bundle. Details will appear in [**2**].

THEOREM 1. *The minimal surfaces of a hypersphere of E^m, the open pieces of a product space of two plane circles and the open pieces of a circular cylinder in E^3 are the only nonminimal surfaces in euclidean space with constant Gauss curvature and mean curvature vector parallel in the normal bundle.*

The proof of this theorem is based on the following lemmas.

LEMMA 1. *Let M be a nonminimal surface in E^m with $\nabla^* \boldsymbol{H} = 0$. Then $M = M_1 \cup M_2$, where M_1 is pseudo-umbilical and $K_N = 0$ on M_2.*

LEMMA 2. *Let M be a nonminimal surface in E^m with $\nabla^* \boldsymbol{H} = 0$, $K_N = 0$ and $K = $ constant, then $K \geqq 0$.*

LEMMA 3 [**3**]. *If M is a nonminimal surface in E^m with $K \geqq 0$ $K_N = 0$ and $\nabla^* \boldsymbol{H} = 0$, then M is one of the following surfaces*; (1) *a minimal surface of a hypersphere of E^m*, (2) *an open piece of a circular cylinder in E^3*, *or* (3) *an open piece of a product space of two plane circles.*

For the normal curvature K_N, we have

THEOREM 2. *The Veronese surface in E^5 is the only closed surface in E^m with constant normal curvature $K_N \neq 0$ and $\nabla^* \boldsymbol{H} = 0$.*

The proof of Theorem 2 is based on Lemma 1 and a result of [**1**] which states that the Veronese surface is the only closed pseudo-umbilical surface in euclidean space with constant normal curvature $K_N \neq 0$.

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