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ON A CLASS OF BEST NONLINEAR APPROXIMATION PROBLEMS

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Let $K(\xi, \eta)$ be a real valued differentiable function defined on $T \times T$ where T is an interval of the real line. More exact requirements on $K(\xi, \eta)$ and T will be specified later. In this paper we report results on existence, uniqueness and characterizations of the best approximation in the L^p norm $(1 \leqq p \leqq \infty)$ to functions of the kind

$$(1) \qquad h(\xi) = \int_\alpha^\beta K(\xi, \eta)\omega(\eta)\, d\eta, \qquad (\alpha, \beta) \subset T,$$

where $\omega(\eta)$ is positive and continuous, by quadrature functions of the form

$$(2) \qquad Q(\xi) = \sum_{i=1}^{r} c_i K(\xi, \eta_i).$$

Here r is fixed, but c_i real and $\eta_i \in T$ are free parameters. The η_i points are called the knots of the approximating functional. More general expressions of the approximation function allow multiple knots and also boundary terms. Specifically, in place of (2), we consider

$$(3) \qquad P(\xi) = \sum_{i=0}^{n-1} a_i \frac{\partial^i}{\partial \eta^i} K(\xi, \alpha) + \sum_{i=0}^{m-1} b_i \frac{\partial^i}{\partial \eta^i} K(\xi, \beta) + \sum_{k=1}^{t} \sum_{l=0}^{\mu_k - 1} c_{kl} \frac{\partial^l K(\xi, \eta_k)}{\partial \eta^{(l)}}$$

where the knots α, β and $\eta_k \in T$ are of multiplicity n, m and μ_k respectively, and the total multiplicity of the interior knots is stipulated to be $\sum_{i=1}^{t} \mu_i = r$. The functions (2) display each η_i as a simple knot with the terms involving the knots α and β omitted.

The class of functions of the form (3) are designated as $\mathscr{S}_{n,m,r}$.

Our main objective is to characterize the best approximation to $h(\xi)$ in the $L^p(T)$ norm from among the functions in $\mathscr{S}_{n,m,r}$. Formally stated, we wish to establish criteria for evaluating $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ and $\{\eta_i\}$ achieving

$$(4) \qquad \min_{a_i, b_i, c_i, \eta_i} \int_T \left| \int_\alpha^\beta K(\xi, \eta)\omega(\eta)\, d\eta - P(\xi) \right|^p d\xi$$

for P of the form (3). We will also investigate the problem of the min

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evaluated with respect to the Q functionals of the form (2). That is, determine c_i and η_i for which

$$(5) \qquad \inf_{c_i, \eta_i} \int_T \left| \int_\alpha^\beta K(\xi, \eta)\omega(\eta)\, d\eta - \sum_{i=1}^r c_i K(\xi, \eta_i) \right|^p d\xi$$

is attained. The existence problem is covered in Theorem 3 below (cf. [**13**, vol. II, p. 63]).

We highlight three important prototypes of this formulation arising in different contexts and motivating our developments.

I. Let $K(\xi, \eta)$ be in $L_2(d\sigma \times d\sigma)$ defined on $T \times T$ where $T = (-\infty, \infty)$. Define

$$G(t, \tau) = \int_T K(t, s)K(\tau, s)\sigma(ds)$$

where σ is a sigma finite measure on T. It is easy to check that G is continuous and positive definite on $T \times T$ and induces a reproducing kernel space. Consider the problem of determining a "best quadrature formula" for the continuous linear functional $L(\tilde{f}) = \int_0^1 \tilde{f}(t)dt$ where $\tilde{f}(t) = \int_T K(t, \tau)f(\tau)\, d\sigma(\tau)$, $f \in L_2(\sigma)$, among the quadrature formulas of the type $Q(\tilde{f}) = \sum_{i=1}^r a_i \tilde{f}(t_i)$, $t_i \in T$.

Let the norm of the functional $\mathscr{R}(\tilde{f}) = L(\tilde{f}) - Q(\tilde{f})$ be denoted by $\|\mathscr{R}_{a,t}\|$ where we indicate the dependence on the parameters $\{a_i, t_i\}_1^r$. The norm is that conjugate to the norm of the reproducing kernel space generated by G. A "best quadrature formula" is delimited as that Q rendering $\|\mathscr{R}_{a,t}\|$ a minimum. The solution to this problem is equivalent to the determination of the best $L_2(\sigma)$ approximation for

$$\inf_{\{a_i, t_i\}} \int_T \left| \int_0^1 K(t, \tau)\, dt - \sum_{i=1}^r a_i K(t_i, \tau) \right|^2 \sigma(d\tau) \qquad \text{(see [13])}.$$

This example has relevance for regression analysis of statistical time series; e.g., see [**11**] and [**12**] and references therein.

II. Consider the class $\mathscr{B}$ of $L_2(D)$ functions analytic in a domain D of the complex plane containing a real segment $[\alpha, \beta]$ with finite norm

$$\|f\|^2 = \iint_D |f(z)|^2 |dz| < \infty.$$

Specify $L(f) = \int_\alpha^\beta \omega(\xi)f(\xi)\, d\xi$ where $\omega(\xi)$ is a continuous positive function on $[\alpha, \beta]$. Consider

$$(6) \qquad \mathscr{Q} = \left\{ Q;\, Q(f) = \sum_{i=1}^r a_i f(\xi_i);\, \xi_i \in D,\ a_i \text{ complex, } r \text{ fixed} \right\}.$$

The space $\mathscr{B}$ has a reproducing kernel $K(z, w)$ on D. It is easy to establish the relationship

$$\inf_{Q \in \mathscr{Q}} \|Q - L\| = \inf_{a_i, \xi_i} \int_D \left| \int_\alpha^\beta \omega(\xi) K(z, \xi)\, d\xi - \sum_{i=1}^r a_i K(z, \xi_i) \right|^2 |dz|$$

e.g., see [**1**], [**2**], [**9**], and [**3**].

III. Define $\mathscr{B} = \{f; f \in C^{n-1}[0, 1], f^{(n-1)}$ absolutely continuous and $f^{(n)} \in L_2[0, 1]\}$ and impose the seminorm $\|f\|^2 = \int_0^1 |f^{(n)}(x)|^2\, dx$ on $\mathscr{B}$. Consider the set of all functionals

$$\mathscr{Q} = \left\{ Q; Q(f) = \sum_{i=1}^r a_i f(\xi_i);\ a_i \text{ real}, 0 < \xi_1 < \xi_2 < \cdots < \xi_r < 1 \right\}.$$

The problem is to determine $Q \in \mathscr{Q}$ achieving

$$\inf_{Q \in \mathscr{Q}} \sup_{\|f\| \leqq 1; f \in \mathscr{B}} \|\mathscr{R}_Q(f)\| \quad \text{where } \mathscr{R}_Q(f) = \int_0^1 f(\xi)\, d\xi - Q(f). \tag{7}$$

It can be shown (e.g., see [**3**]) that the problem of (7) is equivalent to ascertaining the best nonlinear approximation according to

$$\inf_{a_i, c_i, \xi_i} \int_0^1 \left| \int_0^1 (x - \xi)_+^{n-1}\, d\xi - \sum_{i=0}^{n-1} a_i x^i - \sum_{i=1}^r c_i (x - \xi_i)_+^{n-1} \right|^2 dx$$

where the a_i, c_i, ξ_i fulfill certain side constraints. Here $K(x, \xi) = (x - \xi)_+^{n-1} = (x - \xi)^{n-1}$ for $x \geqq \xi$, $=0$ for $x < \xi$.

We postulate henceforth, unless stated to the contrary, that $K(\xi, \eta)$ is extended totally positive (ETP) on $T \times T$; i.e., for any selection of $\xi_i \in T$ and $\eta_j \in T$ satisfying $\xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_p$ and $\eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_p$ (p arbitrary), the compound Fredholm kernel satisfies

$$K_{[p]}(\xi, \eta) = \det \|K^*(\xi_i, \eta_j)\| > 0, \tag{8}$$

with the convention that for groups of equal ξ values (say e.g., $\xi_1 = \xi_2$) the second row in the determinant of (8) is replaced by $\{(\partial K/\partial \xi)(\xi_1, \eta_1), (\partial K/\partial \xi)(\xi_1, \eta_2) \cdots (\partial K/\partial \xi)(\xi_1, \eta_p)\}$ and correspondingly with larger blocks of coincident values higher derivatives are inserted. For coalesced η values the analogous partial derivatives in the second variable appear. The concept of extended total positivity is stronger than that of strict total positivity which insists on strict inequality in (8) only when the ξ's and η's are distinct. The assumption of total positivity has wide scope in analysis and applications; e.g., see [**4**].

With reference to Example II, the reproducing kernel of a simply connected region D symmetric with respect to the real line is indeed ETP as a function of ξ, η varying on $\boldsymbol{R} \cap D$ ($\boldsymbol{R}$ = real line (see [**9**])). The kernel of

Example III, $\Phi(\xi, \eta) = (\xi - \eta)_+^{n-1}$, is totally positive but not ETP. Nevertheless, most of the results announced below extend to this case with the proofs requiring more intricate care.

The first result, of independent interest, essential for our treatment of the minimizing problem (4) delineates bounds on the number of zeros of the function

$$g(x) = \int_\alpha^\beta K(x, \xi)\omega(\xi)\, d\xi - P(x) \tag{9}$$

where $P(x)$ is of the form (3). The notation $Z(g; T)$ denotes the number of zeros of $g(x)$ on T, counting multiplicities. The expression (9) is referred to as an *extended monospline* (abbreviated (EM)). Here $\omega(\xi)$ is continuous and positive on $[\alpha, \beta]$.

THEOREM 1. *Let $K(x, \xi)$ be ETP on $T \times T$ (see* (8)) *and let $[\alpha, \beta]$ be an interval properly contained in T. Then*

$$Z(g; T) \leqq m + n + \sum_{i=1}^{t} (\mu_i + 1) - E \tag{10}$$

where E is the number of knots among $(\eta_1, \eta_2, \ldots, \eta_t)$ of even multiplicity or which lie exterior to the open interval (α, β). In the special case where P in (3) *has the knots at α and β deleted, then the bound in* (10) *is diminished by $m + n$.*

An *extended monospline* $g(x)$ is said to have *full multiplicity* if equality prevails in (10). The following converse theorem is available.

THEOREM 2. *Let $K(x, \xi)$ be ETP on $T \times T$. Let m, n and $\mu_i, i = 1, 2, \ldots, t$, be prescribed and stipulate each μ_i to be odd. Let $x_1 \leqq x_2 \leqq \cdots \leqq x_\gamma$, all $x_i \in T$, be given with $\gamma = m + n + \sum_{i=1}^{t} (\mu_i + 1)$. The number of repeated x values indicates the multiplicity of that x. Then there exists a unique extended monospline $g(x)$ of the form* (9); *i.e., a_i, b_i, c_{kl} and $\eta_1 < \eta_2 < \cdots < \eta_t$, with $\eta_i \in (\alpha, \beta)$ exist which define such that $g(x)$ vanishes precisely on the set $\Gamma = \{x_i\}_1^\gamma$.*

This result incorporates a version for extended monosplines of the fundamental theorem of algebra for monosplines set forth in [7]. Since we require here that $K(x, \xi)$ is ETP, the case of [7] is not subsumed in the present context. In the special case where $m = n = 0$ and each $\mu_i = 1$, the proof of Theorem 2 can be accomplished by appeal to the existence of principal representations for "moment points" occurring in moment spaces of Tchebycheff systems (see [**8**, Chapter 2]). In this connection when $\mu_i = 1$ the coefficients of $K(x, \xi_i)$ in (2) for an extended monospline with zero set of full multiplicity are positive. The proof for the general case of Theorem 2 is quite elaborate involving a continuity technique and the implicit function theorem. The result of Theorem 2 with multiple knots is due to A. Pinkus and this author.

A formulation of Theorems 1 and 2 in the presence of additional boundary constraints satisfied by $g(x)$ is also available. For the case of polynomial monosplines with boundary conditions we refer to [**5**] and [**6**].

We now state the basic characterization for the approximation problem of (4).

THEOREM 3. *Let $K(\xi, \eta)$ be ETP on $T \times T$ and $(\alpha, \beta) \subset T$ properly. Let $1 \leqq p \leqq \infty$ and let σ be a sigma finite measure with an infinite spectrum.*

(i) *Then*

$$\inf_{a_i, b_i, c_i, \xi_i} \left[\int_T \left| \int_\alpha^\beta K(\xi, \eta)\omega(\eta)\, d\eta - P(\xi) \right|^p \sigma(d\xi) \right]^{1/p} \tag{11}$$

is achieved for some $P^* \in \mathscr{S}_{m,n,r}$ with each $\mu_i = 1$ (i.e., with all knots simple) so that

$$P^*(\xi) = \sum_{i=0}^{n-1} a_i^* \frac{\partial^i K}{\partial \eta^i}(\xi, \alpha) + \sum_{i=0}^{m-1} b_i^* \frac{\partial^i K}{\partial \eta^i}(\xi, \beta) + \sum_{i=1}^{r} c_i^* K(\xi, \eta_i^*).$$

Moreover, $\eta_i \in (\alpha, \beta)$, $c_i^* > 0$ and

$$g(x) = \int_\alpha^\beta K(x, \eta)\, d\eta - P^*(x) \tag{12}$$

is an extended monospline with a zero set of full multiplicity.

(ii) *For $p = 2$, the parameters determining P^* may be calculated such that, where $L(x, y) = \int_T K(x, \xi) K(y, \xi) \sigma(d\xi)$,*

$$h(x) = \int_\alpha^\beta L(x, y)\omega(y)\, dy - \sum_{i=0}^{n-1} a_i^* \frac{\partial^i L}{\partial \eta^i}(x, \alpha) - \sum_{i=0}^{m-1} b_i^* \frac{\partial^i L}{\partial \eta^i}(x, \beta) - \sum_{i=1}^{r} c_i^* L(x, \eta_i^*)$$

satisfies

$$h(\alpha) = h'(\alpha) = \cdots = h^{(n-1)}(\alpha) = 0, \qquad h(\beta) = h'(\beta) = \cdots = h^{(m-1)}(\beta) = 0,$$
$$h(\eta_i^*) = h'(\eta_i^*) = 0, \qquad i = 1, 2, \ldots, r. \tag{13}$$

(iii) *For $p = \infty$, the best approximation is characterized uniquely by the equi-oscillation property that $\|g(x)\|_\infty \leqq c$ and there exists $k = 2r + m + n$ points $x_1^0 < x_2^0 < \cdots < x_k^0$ $(\alpha \leqq x_1^0, x_k^0 \leqq \beta)$ satisfying $g(x_i^0)g(x_{i+1}^0) < 0$ and $|g(x_i^0)| = c$, $i = 1, 2, \ldots, k$.*

The question of uniqueness for the minimizing P^* in (11) (except for the L_∞ case) is unsettled.

A key tool in the proof of Theorem 3 and of intrinsic basic importance is what we call the improvement theorem.

THEOREM 4 (IMPROVEMENT THEOREM). *Let $g(x)$ be an extended monospline (EM) where each μ_i is odd and suppose $g(x)$ has a zero set Γ of full multiplicity so that*

$$Z(g(x); T) = m + n + \sum_{i=1}^{t} (\mu_i + 1) = m + n + 2l.$$

Then there exists a unique EM

$$\tilde{g}(x) = \int_{\alpha}^{\beta} K(x, \xi)\omega(\xi)\, d\xi - \tilde{P}(x)$$

where $\tilde{P}$ vanishes precisely on Γ, exhibits l simple knots and compared to $g(x)$ satisfies the global inequality

$$|\tilde{g}(x)| \leqq |g(x)|, \quad \text{for all } x \in T, \tag{14}$$

with equality prevailing everywhere iff $\tilde{g}(x) \equiv g(x)$.

The assertion of Theorem 4 says in essence that in most norms a multiple knot quadrature approximation for extended monosplines can always be improved by a simple knot approximation. The result of Theorem 4 also applies for ordinary polynomial monosplines.

Detailed proofs of these and related theorems concerning orders of approximation as $r \to \infty$ with application of these results to the models of I–III will be published elsewhere.

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