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ON THE TYPE OF ASSOCIATIVE *H*-SPACES

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Let X denote a connected, simply connected associative H-space of finite dimension and type which is not acyclic. By a well-known theorem of Hopf, $H_*(X; Q)$ is an exterior algebra on generators of odd degree, $E(x_1, x_2, \ldots, x_k)$. We define the rank of X to be the number of generators, k, and the type of X to be the k-tuple $[n_1, n_2, \ldots, n_k]$ where $n_i = \deg x_i$ and, for convenience, $n_i \leqq n_{i+1}$.

There are two kinds of questions one may ask. First, given that the rank of X is some fixed small number, what possible types may occur? Results for rank $X \leqq 4$ can be found in [**2**], [**3**], and [**4**]. Secondly, one may ask what possible types can occur if the greatest entry in the type of X is known. In particular, if the greatest entry is large are there associative H-spaces whose types continue the sequence of types of exceptional Lie groups as has been conjectured? The present paper gives some answers to both kinds of questions. The main results are the following:

THEOREM 1. *If* rank $X \leqq 5$ *then either* X *has the type of a Lie group or has one of the types* $[3, 5, 7, 11, 15]$, $[3, 7, 11, 11, 15]$ *or* $[3, 5, 5, 7, 9]$.

THEOREM 2. *Assume that the type of* X *has no entry occurring more than once and that the greatest entry,* $2n - 1$, *is greater than* 59.

(i) *If* n *is odd then the type of* X *has all the numbers* $\{2i - 1 | 2 \leqq i \leqq n\}$ *among its entries.*

(ii) *If* n *is even and* $n/2$ *is not a power of* 3 *then the type of* X *has all numbers* $\{4i - 1 | 1 \leqq i \leqq n/2\}$ *among its entries.*

(iii) *If* n *is even and* $n/2$ *is a power of* 3 *then the type of* X *has all numbers* $\{4i - 1 | 1 \leqq i \leqq (n/2) - 1\}$ *among its entries.*

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The proofs use the classifying space of X, BX, and a generalization of results proved by Clark in [**1**]. Since X has finite dimension and type it can have torsion for only a finite number of primes. If $H_*(X;\mathbf{Q}) = E(x_1, x_2, \ldots, x_k)$ with $\deg x_i = n_i$ and X has no p-torsion then by a generalization of Borel's transgression theorem $H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra on k generators, $y_1, y_2, \ldots, y_k$, where $\deg y_i = n_i + 1$. Therefore $H^*(BX;\mathbf{Z}_p)$ has this form for all but a finite number of primes. We have the following theorem concerning such cohomology algebras:

THEOREM 3. *Let Y be a space such that $H^*(Y;\mathbf{Z}_p)$ is a polynomial algebra on generators of even degree. If $H^*(Y;\mathbf{Z}_p)$ has k generators $x_1, x_2, \ldots, x_k$ where $\deg x_i = 2m_i$ then $H^*(Y;\mathbf{Z}_p)$ has k generators $y_1, y_2, \ldots, y_k$ where $\deg y_i \equiv 1 - p \bmod (m_1, m_2, \ldots, m_k)$ or else some $m_i \equiv 0 \bmod p$. (The symbol $(m_1, m_2, \ldots, m_k)$ represents the greatest common divisor of $m_1, m_2, \ldots, m_k$.)*

The proof of Theorem 3 consists of looking at the action of the mod p Steenrod algebra on $H^*(Y;\mathbf{Z}_p)$ and reducing the problem to one concerning the heights of prime ideals in a certain subalgebra of $H^*(Y;\mathbf{Z}_p)$.

By using a theorem of Dirichlet (that if $(m, n) = 1$ then there are an infinite number of primes $\equiv m \bmod n$) we can "translate" the theorem above into one concerning the type of X.

THEOREM 4. *If $2m_1 - 1, 2m_2 - 1, \ldots, 2m_s - 1$ are entries in the type of X and $k - 1$ is relatively prime to $(m_1, m_2, \ldots, m_s)$ then there are s entries in the type of X; $2n_1 - 1, 2n_2 - 1, \ldots, 2n_s - 1$; such that $n_i \equiv k \bmod (m_1, m_2, \ldots, m_s)$ for $i = 1, 2, \ldots, s$.*

By considering special cases we can obtain many corollaries of Theorem 4, most of them highly technical. The following two deserve special mention.

COROLLARY 5. *If $2m - 1$ is the greatest entry in the type of X and $2m - 1$ occurs as an entry s times then for each k with $1 < k < m$ and $(k - 1, m) = 1$ we can conclude that $2k - 1$ occurs as an entry at least s times.*

REMARK. For $k = 2$, Corollary 5 states that if the greatest entry occurs s times then 3 occurs as an entry at least s times.

COROLLARY 6. *If $2m - 1$ is any entry of the type of X occurring s times then $s\phi(m) \leqq \operatorname{rank} X$, where ϕ is the Euler toitient function.*

By systematically applying Theorem 4 and its corollaries the proof of Theorem 1 follows. Theorem 2 is proved by using a large amount of elementary number theory and Theorem 4, with careful attention to the congruence class of the greatest entry mod 3 and 5.

It should be remarked that although our proof of Theorem 2 in general requires the assumption that each entry occurs only once, there are many results obtainable without this restriction. One such easily stated result is the following:

THEOREM 7. *If* $2n - 1$ *is the greatest entry in the type of* X *and* n *is odd then the type of* X *has all numbers* $\{4i - 1 | 1 \leqq i \leqq (n - 1)/2\}$ *among its entries.*

Finally we make a remark concerning the technique. The technique employed here essentially consists of proving a result about algebras over the Steenrod algebra and then applying elementary number theory. The elementary number theory being somewhat exhausted, any improvement in results seems to lie in better results about algebras over the Steenrod algebra. In some sense Theorem 3 of the present paper uses only the simplest Adem relations. A more detailed analysis of algebras over the Steenrod algebra would necessarily take into account the higher Adem relations. In this context the use of secondary operations is naturally introduced, from which, hopefully, more precise information will follow.

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