BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

AN EXACT SEQUENCE INVOLVING THE CHERN CHARACTER

BY ALBERT T. LUNDELL[1]

Communicated by P. E. Thomas, April 28, 1971

In [3] the author defined maps $b_n':U(n)\to\Omega^2U(n+1)$ which are deformations of the classical Bott homotopy equivalence $b:U\to\Omega^2U$ [1], i.e., the composite $U(n)\to\Omega^2U(n+1)\to\Omega^2U$ is homotopic to the composite $U(n)\to U\to\Omega^2U$. The maps b_n' are natural with respect to the inclusions $U(k)\subset U(n)$ for $k\leqq n$. The maps b_n' may be used to define homomorphisms $B_n:\pi_r(U(n))\to\pi_{r+2}(U(n+1))$ as the composite homomorphism

$$\pi_r(U(n))\xrightarrow{b'_{n*}}\pi_r(\Omega^2U(n+1))\xrightarrow{\partial^{-2}}\pi_{r+2}(U(n+1)).$$

The advantage gained by using the maps B_n is that they give information on the nonstable homotopy of $U(n)$ not available from the classical Bott maps, and they agree with the classical results in the stable range. For example, the results of [3] show that the map $B_n:\pi_r(U(n))\to\pi_{r+2}(U(n+1))$ is an isomorphism for $r\leqq 2n-1$, and $B_n:\pi_{2n}(U(n))\to\pi_{2(n+1)}(U(n+1))$ is a monomorphism. Kenneth Millett has calculated $B_n:\pi_{2(n+r)}(U(n))\to\pi_{2(n+r+1)}(U(n+1))$ for $r=2,\ 3$.

The purpose of this announcement is to describe an application of the maps b_n' to complex K-theory. We work throughout in the category of finite CW complexes with basepoint. We use Q to denote the additive group of rational numbers, and Z to denote the group of integers.

1. **The spectrum TU.** We use the maps b_n' to define a spectrum TU by setting $TU_{2k}=\Omega U(k)$, $TU_{2k+1}=U(k)$ for $k\geqq 0$, and $TU_m=\text{point}$ for $m<0$. The maps of the spectrum are

$$\tau_{2k}=\text{id}:TU_{2k}=\Omega U(k)\to\Omega U(k)=\Omega TU_{2k+1}$$

and

$$\tau_{2k+1}=b_k':TU_{2k+1}=U(k)\to\Omega^2U(k+1)=\Omega TU_{2k+2}.$$

AMS 1970 *subject classifications*. Primary 55B20; Secondary 55B15.

Key words and phrases. Chern character, K-theory, Bott map, generalized cohomology, spectrum.

[1] This work was partially supported by NSF Grant GP8965.

We call $\boldsymbol{TU}$ the *nonstable unitary spectrum.*

To compute the homotopy of this spectrum, we use the maps $B_n: \pi_r(U(n)) \to \pi_{r+2}(U(n+1))$ mentioned above and results about the iterates of these maps.

THEOREM 1. *The homotopy groups of* $\boldsymbol{TU}$ *are as follows*

$$\begin{aligned} \pi_{2s-1}(\boldsymbol{TU}) &\cong Q/Z \quad \text{for } s \geqq 0, \\ \pi_{2s}(\boldsymbol{TU}) &\cong Z \quad \text{for } s < 0, \\ \pi_r(\boldsymbol{TU}) &= 0 \quad \text{otherwise.} \ \square \end{aligned}$$

2. **Relative spectra and cohomology.** A relative spectrum $(\boldsymbol{E}, \boldsymbol{F})$ consists of spectra $\boldsymbol{E}$ and $\boldsymbol{F}$ and inclusion maps $F_k \subset E_k$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc} F_k & \xrightarrow{\epsilon_k'} & F_{k+1} \\ \cap\downarrow & & \downarrow\cap \\ E_k & \xrightarrow{\epsilon_k} & E_{k+1}. \end{array}$$

By using the co-exact sequence of pairs

$$(X, *) \to (X, X) \to (CX, X) \to (SX, *) \to (SX, SX) \to \cdots$$

and taking direct limits of the exact sequences of homotopy sets

$$\cdots \to [S^{k+1-n}X; E_k] \to [CS^{k-n}X, S^{k-n}X; E_k, F_k] \to [S^{k-n}X; F_k] \to [S^{k-n}X; E_k] \to \cdots$$

where $(\boldsymbol{E}, \boldsymbol{F})$ is a relative spectrum, one obtains an exact cohomology sequence similar to the ordinary cohomology coefficient sequence.

THEOREM 2. *If* $(\boldsymbol{E}, \boldsymbol{F})$ *is a relative spectrum, there is a long exact sequence* $(-\infty < n < \infty)$,

$$\cdots \to h^{n-1}(X; \boldsymbol{E}) \xrightarrow{j_*} h^{n-1}(X; \boldsymbol{E}, \boldsymbol{F}) \xrightarrow{\beta} h^n(X; \boldsymbol{F}) \xrightarrow{i_*} h^n(X; \boldsymbol{E}) \to \cdots. \ \square$$

The unitary spectrum $\boldsymbol{BU}$ is defined by $\boldsymbol{BU}_{2k} = \Omega U$ and $\boldsymbol{BU}_{2k+1} = U$ with maps $\mathrm{id}: \boldsymbol{BU}_{2k} = \Omega U \to \Omega U = \Omega \boldsymbol{BU}_{2k+1}$ and $b: \boldsymbol{BU}_{2k+1} = U \to \Omega^2 U = \Omega \boldsymbol{BU}_{2k+2}$, where b is the classical Bott homotopy equivalence. We easily check that $(\boldsymbol{BU}, \boldsymbol{TU})$ is a relative spectrum and thus obtain

COROLLARY 3. *There is a long exact cohomology sequence*

$$\cdots \to h^{n-1}(X; \boldsymbol{BU}) \xrightarrow{j_*} h^{n-1}(X; \boldsymbol{BU}, \boldsymbol{TU}) \xrightarrow{\beta} h^n(X; \boldsymbol{TU}) \xrightarrow{i_*} h^n(X; \boldsymbol{BU}) \to \cdots. \ \square$$

Homotopy groups of a relative spectrum $(\boldsymbol{E}, \boldsymbol{F})$ are defined in the usual way, and there is a long exact sequence of homotopy groups of spectra involving the homotopy groups $\pi_n(\boldsymbol{E}, \boldsymbol{F})$.

For the relative spectrum $(\boldsymbol{BU}, \boldsymbol{TU})$, a calculation shows that the following result holds.

THEOREM 4. *The homotopy groups of* $(\boldsymbol{BU}, \boldsymbol{TU})$ *are as follows*

$$\pi_{2s}(\boldsymbol{BU}, \boldsymbol{TU}) \cong Q \quad \text{for } s \geqq 0,$$
$$\pi_r(\boldsymbol{BU}, \boldsymbol{TU}) = 0 \quad \text{otherwise.}$$

Moreover, the exact sequences $0 \to \pi_{2s}(\boldsymbol{BU}) \to \pi_{2s}(\boldsymbol{BU}, \boldsymbol{TU}) \to \pi_{2s-1}(\boldsymbol{TU}) \to 0$ *for* $s \geqq 0$ *and* $0 \to \pi_{2s}(\boldsymbol{TU}) \to \pi_{2s}(\boldsymbol{BU}) \to 0$ *for* $s < 0$ *are just the exact sequences* $0 \to Z \to Q \xrightarrow{\rho} Q/Z \to 0$ *and* $0 \to Z \to Z \to 0$, *respectively.* □

3. **Interpretation of the exact cohomology sequence.** In the cohomology sequence, the terms $h^n(X; \boldsymbol{BU}) \cong \tilde{K}^n(X)$, reduced complex K-theory. (Recall we are working in the category of based complexes.)

The groups $h^n(X; \boldsymbol{BU}, \boldsymbol{TU})$ and the maps $j_*: h^n(X; \boldsymbol{BU}) \to h^n(X; \boldsymbol{BU}, \boldsymbol{TU})$ are determined by the following proposition. Let H^* denote ordinary singular cohomology.

PROPOSITION 5. *For each* n, $h^n(X; \boldsymbol{BU}, \boldsymbol{TU}) \cong \sum_{r \geq 0} \tilde{H}^{n+2r}(X; Q)$. *The map* j_* *is the truncated (from below) Chern character*

$$j_* = ch' = \sum_{r \geq 0} ch_{n+2r}: \tilde{K}^n(X) \to \sum_{r \geq 0} \tilde{H}^{n+2r}(X; Q). \quad \square$$

Note that for $n \leqq 1$, $ch' = ch$, the Chern character. This proposition is proved using theorems about generalized cohomology theories derived from the spectral sequence. See Dyer [2, Chapter I].

Let $T^n(X) = h^n(X; \boldsymbol{TU})$. It remains to analyze $T^n(X)$ and the maps $i_*: T^n(X) \to \tilde{K}^n(X)$, $\beta: h^n(X; \boldsymbol{BU}, \boldsymbol{TU}) \to T^{n+1}(X)$. An easy calculation with the Chern character and the truncated Chern character establishes

PROPOSITION 6. *For each* n, $T^{n+1}(X) = \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes Q/Z + \tilde{T}^{n+1}(X)$. *The map* β *has cokernel isomorphic to* $\sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes Q/Z$. *The map* i_* *is a monomorphism* $\tilde{T}^{n+1}(X) \to \tilde{K}^{n+1}(X)$. □

REMARKS. (i) Although $\operatorname{Im} \beta \cong \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes Q/Z$, the map β is not $1 \otimes \rho: \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes Q \to \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes Q/Z$ where ρ is the projection $Q \to Q/Z$.

(ii) The direct sum splitting of $T^n(X)$ is not natural with respect to maps $f: X \to Y$, as can be seen by using the inclusion map $RP^{2n-1} \to RP^{2n}$.

(iii) The long exact cohomology sequence decomposes into exact sequences of length four

$$0 \to \tilde{T}^n(X) \to \tilde{K}^n(X) \xrightarrow{ch'} \sum_{r \geqq 0} \tilde{H}^{n+2r}(X; Q) \to \sum_{r \geqq 0} \tilde{H}^{n+2r}(X) \otimes Q/Z \to 0$$

although this is not a natural decomposition.

Let tors(G) denote the torsion subgroup of G. An analysis of $\tilde{T}^n(X)$ yields

PROPOSITION 7. (i) *For* $n \leqq 1$, $\tilde{T}^n(X) \cong \text{tors}(\tilde{K}^n(X))$.

(ii) *For* $1 < n \leqq \dim X$,

$$\tilde{T}^n(X) \cong \text{tors}(\tilde{K}^n(X)) \oplus \sum_{r<0} \tilde{H}^{n+2r}(X)/\text{tors}(\tilde{H}^{n+2r}(X)).$$

(iii) *For* $\dim X < n$, $\tilde{T}^n(X) \cong \tilde{K}^n(X)$. □

In this proposition, $\dim X$ may be interpreted as the rational singular cohomological dimension.

The preceding propositions are collected in the following

THEOREM 8. *For each finite* CW *complex* X, *there is a long exact sequence* $(-\infty < n < \infty)$,

$$\cdots \to \sum_{r \geqq 0} \tilde{H}^{n-1+2r}(X) \otimes Q/Z \oplus \tilde{T}^n(X) \to \tilde{K}^n(X) \xrightarrow{ch'} \sum_{r \geqq 0} \tilde{H}^{n+2r}(X; Q)$$
$$\to \sum_{r \geqq 0} \tilde{H}^{n+2r}(X) \otimes Q/Z \oplus \tilde{T}^{n+1}(X) \to \tilde{K}^{n+1}(X) \to \cdots. \quad \square$$

Detailed proofs, applications, and extensions of these results will appear elsewhere.

BIBLIOGRAPHY

1. R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337. MR **22** #987.

2. E. Dyer, *Cohomology theories*, Mathematics Lecture Note Series, Benjamin, New York, 1969.

3. A. T. Lundell, *A Bott map for non-stable homotopy of the unitary group*, Topology **8** (1969), 209–217. MR **38** #6595.

UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302