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ORDERS IN SEMILOCAL RINGS[1]

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A semilocal ring S is one such that $S/\text{rad } S$ is semisimple (= a direct sum of simple modules). The first theorem generalizes a theorem of Faith and Utumi [**65**] to semilocal rings, which had been extended by Robson [**67**] to artinian rings. (Cf. also Procesi [**65**].)

(A) THEOREM. *If R has a semilocal right quoring* (= *quotient ring*) $S=D_n$ *which is a full* $n\times n$ *matrix ring, then there exists a set* M *of* $n\times n$ *matrix units such that* R *contains a right order* F_n *of* D_n, *and* F *is a right order in* $D=\text{centralizer } M$.

The proof depends on the lemma.

(B) LEMMA. *If* F_n *has semilocal right quoring* D_n, *where* F *is subring of* D, *then for every regular element* $t\in F_n$, *there exists a regular element* $a\in F_n$ *such that* $x=ta=(x_{ij})$ *is represented by a matrix* (x_{ij}) *with diagonal elements* x_{ii} *regular elements of* F, *and the off-diagonal elements* x_{ij} *are contained in* $F\cap\text{rad } D$.

This was proved by Faith-Utumi [**65**] for semisimple D, and Robson [**67**] for artinian D. The proof of (A) makes use only of the case D is semisimple, and both (B) and (A) require the rather obvious fact that if R has semilocal right quoring S, then $\overline{R}$ has a semisimple right quoring $\overline{S}=S/\text{rad } S$, where $\overline{R}\approx R/(R\cap\text{rad } S)$ is the image of R under the canonical map $S\to\overline{S}$. Then, $\overline{R}$ is semiprime right Goldie.

(A) has the following application.

(C) THEOREM. *Any maximal* $\sim_l$ *right order in a semilocal ring* $S=D_n$ *is isomorphic to the endomorphism ring of a torsion-free unital module over a right order of* D.

The proof is patterned after that of Faith [**64**] for the case D is a field, and Robson [**67**] for D artinian. (*Note.* Two right orders R_1 and

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R_2 are left equivalent, notation $R_1 \sim_l R_2$, provided that $R_1 b \subseteq R_2$ and $R_2 a \subseteq R_1$ for two regular elements a and b of S.)

Any right self-injective semilocal ring D is a quotient ring, a fact that shows that (A) holds for rings more general than D artinian. Thus, (A) holds for a ring which is right PF in the sense of Azumaya [66] (cf. Kato [68], Onodera [68], Osofsky [66], and Utumi [67]).

An ideal T is *reflective* if and only if

$$c \in R \text{ is regular} \Leftrightarrow [c + T] \text{ is regular in } R/T.$$

A right ideal I is a *q-regular right ideal* if $c+x$ is regular in R for all regular $c \in R$, and all $x \in I$. Every reflective ideal is q-regular. An ideal T is *right quorite* provided that for every regular $c \in R$, and every $x \in T$, there is a regular $c_1 \in R$, and an element $x_1 \in T$ such that $xc_1 = cx_1$. (*Mnemonic.* $c^{-1}x = x_1 c_1^{-1}$.) If R has a right quoring S, then every closed ideal of R is right quorite. An ideal T is (*semiprime*) *right Goldie* provided that R/T is (*semiprime*) right Goldie.

(D) THEOREM. *A ring R has a semilocal right quoring if and only if the sum T of all the q-regular right ideals of R is a reflective, right quorite, and right semiprime Goldie ideal.*

The proof depends upon the theorem of Feller and Swokowski [61] which asserts that R has a right quoring whenever R has a reflective, right quorite S ideal T such that R/T has a right quoring Q, and in this case $TS = STS$, and $TS \cap R = T$. Furthermore, it is clear that every reflective ideal is q-regular. Therefore, *the condition of Theorem* (D) *is the assertion that there exists a maximal q-regular ideal, reflective, quorite ideal T which contains every q-regular right ideal.* This is reminiscent of (and indeed, q-regularity generalizes) the Perlis-Jacobson characterization of the radical of a ring.

(E) COROLLARY. *A ring R has a semilocal right quoring S which is:* (a) *right noetherian,* (b) *right artinian,* (c) *a ring with right vanishing radical,* (d) *semiprimary if and only if R contains an ideal T with the properties stated in* (D) *and with the respective properties*;

(a) *R satisfies the a.c.c. on closed right ideals.*

(b) *T is nilpotent, and R satisfies the a.c.c. on closed right ideals.* (*Then R satisfies the d.c.c. on closed right ideals, and conversely.*)

(c) *T is right vanishing* (*=right T-nilpotent*).

(d) *T is nilpotent.*

In the case (c), then R is left perfect by Bass's theorem [60], so this characterizes rings with perfect right quotient rings. Cf. Jategaonkar [69], Gupta [68], Gupta and Saha [67], Robson [67], Small [66], and Talintyre [66]. For other pertinent references to classical quotient rings, including those which are quasi-Frobenius, consult Elizarov [69].

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