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HOLOMORPHIC APPROXIMATION ON TOTALLY REAL SUBMANIFOLDS OF A COMPLEX MANIFOLD[1]

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1. **Introduction.** Let M be a real differentiable submanifold of a complex manifold X of differentiability class C^k, $1 \leqq k \leqq \infty$. Letting $T_x(M)$ and $T_x(X)$ denote the tangent spaces to M and X at $x \in M$, we see that $T_x(M)$ is a real linear subspace of the complex vector space $T_x(X)$. We say that M is a *totally real submanifold* of X if for each point $x \in M$, $T_x(M)$ contains no nonzero complex subspaces of $T_x(X)$.

There are many examples of totally real submanifolds (see, e.g. Nirenberg-Wells [**8**]), the simplest examples being a real curve in X, the distinguished boundary of a polydomain, or $\boldsymbol{R}^n \subset \boldsymbol{C}^n$. The geometric nature of a totally real submanifold implies that it behaves in many cases like this last example. In particular, the Weierstrass approximation theorem tells us that holomorphic functions on $\boldsymbol{C}^n$ are dense in the Banach space of continuous functions on a compact subset $K \subset \boldsymbol{R}^n \subset \boldsymbol{C}^n$ in the supremum norm. There have been various investigations recently generalizing this type of theorem to compact subsets of totally real submanifolds (see Čirka [**1**], Hörmander-Wermer [**6**], Nirenberg-Wells [**8**]). In §2 we formulate our main results on holomorphic approximation, in which we improve on the previous known results by (a) extending the domain of definition of the approximating functions, (b) minimizing the differentiability requirements for the submanifold, and (c) requiring that the approximation be uniform on K along with uniform approximation of all derivatives up to the order of differentiability of the submanifold. In §3 we formulate sheaf injection theorems for hyperfunctions on a totally real submanifold of a complex manifold. For instance, Theorem 4.1 says that the sheaf of germs of distributions is canonically embedded in the sheaf of germs of hyperfunctions on a C^∞ totally real submanifold of a complex manifold (cf. Martineau [**7**] and Harvey [**4**]).

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Proofs of our results are discussed only briefly; details will appear elsewhere.

2. **Holomorphic approximation theorems.** Let X denote throughout this paper a paracompact complex manifold of complex dimension n. Suppose K is compact in U which is open in X. We denote by $\mathcal{O}(U)$ the Fréchet space of holomorphic functions on U, $\mathcal{O}(K)$ the holomorphic functions on K (inductive limit of Fréchet spaces $\mathcal{O}(U)$, for U open and $U \supset K$ with the inductive limit topology), $C^k(U)$ the complex-valued functions on U which are k times continuously differentiable, $C^k(K)$ the Banach space of limits of C^∞ functions defined near K in the C^k-norm, $0 \leqq k \leqq \infty$. Let $C^0(K) = C(K)$. The C^k-norm on K, for k finite, is defined by $\|f\|_{K,k} = \sup_{x\in K;\, |\alpha| \leqq k} |D^\alpha f(x)|$, the derivatives being defined locally, and the norm being defined globally by means of a partition of unity. Our principal result is the following theorem.

THEOREM 2.1. *Let M be a C^k totally real submanifold of X, where $1 \leqq k \leqq \infty$, and let K be a compact subset of M. Then there exists an open Stein neighborhood U of M in X such that the natural restriction mapping*

$$\mathcal{O}(U) \to C^{k-1}(K)$$

has dense range.

In particular, if M is C^1 then we have the

COROLLARY 2.2. *$\mathcal{O}(U)$ is dense in $C(K)$.*

The proof of Theorem 2.1 breaks down into two parts: Part I, showing that there is a $U \supset M$ so that $\mathcal{O}(U) \to \mathcal{O}(K)$ is dense; Part II, showing that $\mathcal{O}(K) \to C^{k-1}(K)$ is dense. Part I follows by the construction of a strongly plurisubharmonic exhaustion function of the appropriate type and standard Runge approximation arguments yielding the following generalization of Grauert's tubular neighborhood results in [2] (the real-analytic case).

PROPOSITION 2.3. *Let M be a C^1 totally real submanifold of X then there exists an open Stein neighborhood U of M in X such that K is $\mathcal{O}(U)$-convex. In particular, K is $\mathcal{O}(M)$-convex.*

Part II is shown by two different methods outlined in §§3 and 4. In previous papers the uniform approximation (Corollary 2.2) was obtained with U a small neighborhood of K, which did not necessarily contain M. That $\mathcal{O}(K)$ has dense image in $C(K)$ was proven in [8] only for $k \geqq 4n-1$ (with similar dimensional restrictions in [6]), whereas we now obtain this result for $k=1$.[2] The techniques we use involve Ramírez integral kernels which invert $\bar{\partial}$ in strongly pseudoconvex domains, giving good uniform estimates (see e.g., Ramírez [9], Grauert-Lieb [3]).

3. **Global holomorphic kernels for the $\bar{\partial}$-operator in $\mathbf{C}^n$.** If M is a C^1 totally real submanifold of $\mathbf{C}^n$, then there exists a C^2 strongly plurisubharmonic function ϕ defined near M whose vanishing defines M. Let $T(\epsilon)=\{x:\phi(x)<\epsilon\}$. We have the following theorem.

THEOREM 3.1. *Let K be a compact subset of a C^k totally real submanifold M of $\mathbf{C}^n$, where $1\leqq k\leqq\infty$. Then there exists a domain of holomorphy U containing K such that if ω is a C^∞ $\bar{\partial}$-closed form of type* (0, 1) *with compact support in $T(\eta)\cap U$ which vanishes to order k on M, then, for each ϵ, with $0<\epsilon<\eta/3$, there exists a function $u_\epsilon\in C^\infty(T(\epsilon)\cap U)$ such that*

$$\bar{\partial}u_\epsilon=\omega \quad \textit{in } T(\epsilon)\cap U,$$

and

$$\|u_\epsilon\|_{K,k}\leqq C_\epsilon\|\omega\|_{\operatorname{supp}\omega,0},$$

where the constants $C_\epsilon\to 0$ as $\epsilon\to 0$.

Theorem 2.1, for $X=\mathbf{C}^n$, now follows easily from Theorem 3.1 combined with Proposition 2.3. The proof of Theorem 3.1 proceeds by constructing a holomorphic kernel $\Omega^\epsilon(\zeta, z)$ defined in the closure of $T(3\epsilon)\cap U$ which is an $(n, n-1)$ form in the ζ variable and holomorphic in z. This form has a singularity at $\zeta=z$ of order $|\zeta-z|^{2n-1}$, and for $\zeta\in\partial T(3\epsilon)$, $\Omega^\epsilon(\zeta, z)$ is holomorphic in z, for $z\in T(\epsilon)\cap U$. Kernels of this type were first constructed by Ramírez [9] in a single strongly pseudoconvex domain. We have modified his construction in a suitable manner to conform to our given geometric problem. The domain $T(\epsilon)\cap U$ is only strongly pseudoconvex on the $T(\epsilon)$ part of the boundary, but this suffices for the construction we carry out. The solution u_ϵ in Theorem 3.1 is defined by

$$u_\epsilon=\int_{T(3\epsilon)\cap U}\Omega^\epsilon\wedge\omega,$$

and the estimates in Theorem 3.1 follow from careful estimates of the L^1-norms of $\Omega^\epsilon(\zeta, z)$, uniform in $z\in K$, keeping track of its dependence on ϵ.

[2] Čirka [1] also states that $\mathcal{O}(K)$ has dense image in $C(K)$ if M is C^1, but there seems to be a gap in his proof.

4. Hyperfunctions on totally real submanifolds. Let $\mathcal{C}^k$ denote the sheaf of germs of C^k-functions on a C^k-manifold M. Let $\mathcal{D}'_k$ denote the sheaf of germs of distributions of order k on M, i.e., $\mathcal{D}'_k(\Omega)$ for an open set Ω is defined to be the dual of the space $C_0^k(\Omega)$ of compactly supported C^k-functions on Ω with the usual inductive limit topology. We let $\mathcal{D}'=\mathcal{D}'_\infty$ and $\mathfrak{M}=\mathcal{D}'_0$, the sheaf of germs of regular Borel measures on M.

Suppose M is a C^k totally real submanifold of a complex manifold X. The *sheaf* $\mathcal{B}$ *of germs of hyperfunctions on* M (of type Ω^n) is, by definition, the sheaf generated by the presheaf $U\to H^n_{U\cap M}(U, \Omega^n)$. Here $H^n_{U\cap M}(U, \Omega^n)$ denotes the nth relative cohomology group of U modulo $U-M$ with coefficients in Ω^n (the sheaf of germs of holomorphic n-forms). Since each point in M has a compact neighborhood $K\subset M$ with $\mathcal{O}(K)$ dense in $C(K)$ (see Theorem 4.2 below) the set M is a "totally real set" in the sense of [5]. Consequently Sato's theory of hyperfunctions, as developed in [5] (Theorem 3.9 and its Corollary 3.10), is valid for M. In particular, since each compact set $K\subset M$ is holomorphically convex, Corollary 3.10 in [5] says that the space $\mathcal{O}(K)'$, of analytic functionals, is isomorphic to $\Gamma_K(M, \mathcal{B})$, the space of hyperfunctions supported in K. The following theorem tells us that a distribution on M is a special case of a hyperfunction.

THEOREM 4.1. *Let M be a C^k totally real submanifold of X, where $1\leqq k\leqq\infty$. Then there is a natural sheaf injection of $\mathcal{D}'_{k-1}$ into $\mathcal{B}$.*

This theorem, which is a purely local result, has as an immediate consequence that, for each compact set $K\subset M$, $\Gamma_K(M, \mathcal{D}'_{k-1})\cong\Gamma(K, \mathcal{C}^{k-1})'$ is injected into $\Gamma_K(M, \mathcal{B})\cong\mathcal{O}(K)'$. By the Hahn-Banach theorem this implies the global approximation Theorem 2.1 (or more correctly that $\mathcal{O}(K)$ is dense in $C^{k-1}(K)$).

The crucial ingredient in the proof of Theorem 4.1 is the following strong local approximation theorem.

THEOREM 4.2. *Let M be a C^k totally real submanifold in $\mathbf{C}^n$. If $p\in M$, then there is a neighborhood U of p so that if $f\in C_0^k(U)$, and K is a compact subset of M with* supp $f\cap K=\varnothing$, *then there exists a sequence f_j, with $f_j\in\mathcal{O}(U\cap M)$, and such that*

(a) *$f_j\to f$ in $C^{k-1}(U\cap M)$,*

(b) *$f_j\to 0$ in $\mathcal{O}(K)$.*

The first approximation in $C^{k-1}(U\cap M)$ is uniform on compact subsets with all derivatives up to order k. In part (b) we construct the $f_j\in\mathcal{O}(M\cap U)$ in a fixed open neighborhood V of K with $f_j\to 0$ uniformly on compact subsets of V. This theorem is proved by setting up and solving a local $\bar\partial$-problem with estimates, similar to the technique described in §3—except more explicit. We find a local family of Cauchy-Fantappiè kernels defined explicitly in terms of Taylor polynomials of a strongly plurisubharmonic function ϕ whose vanishing defines M.

REMARK. This second proof of Theorem 2.1 using hyperfunctions avoids the use of the global holomorphic kernels of §3, and is valid on arbitrary complex manifolds. The global kernel is replaced by a localization technique (Theorem 1.3 in [5]) borrowed from Sato's theory of hyperfunctions.

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