BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 5, September 1971

THE FINITENESS OF I WHEN $R[X]/I$ IS FLAT

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Communicated by Hyman Bass, April 2, 1971

Let R be a commutative ring with identity, let X be a single indeterminate, and let I be an ideal of $R[X]$. Denote by min I the set $\{f \neq 0 \in I \mid \deg f \leqq \deg g \text{ for all } g \neq 0 \in I\}$. Let $c(I)$ denote the ideal of R generated by the coefficients of the elements of I. We use $\bar{R}$ for the integral closure of R (in its total quotient ring) and $J(R)$ for the intersection of the maximal ideals of R. By a regular element, we mean a nonzero-divisor. An R-module M is called torsion-free if $rm = 0$, $r \in R$, $m \neq 0 \in M$, implies r is a zero-divisor of R.

1. Main results. (Proofs and details will appear elsewhere.) We assume throughout this section that min I contains a regular element of $R[X]$.

1.1 THEOREM. *If $R[X]/I$ is a flat R-module, then I is a finitely generated ideal of $R[X]$.*

The proof proceeds as follows. First prove the theorem in the case that R is quasi-local integrally closed with infinite residue field. Then remove the infinite residue field assumption by adjoining an indeterminate. Next remove the quasi-local assumption, and finally remove the assumption that R be integrally closed.

If R is integrally closed, the generators of I in 1.1 can be chosen from min I. In proving 1.1, we obtain the following more precise result in the case that R is quasi-local integrally closed.

1.2 THEOREM. *If R is quasi-local integrally closed, then the following are equivalent:*

(i) *I is principal and $c(I) = R$.*

(ii) *$R[X]/I$ is R-flat.*

(iii) *$R[X]/I$ is R-torsion-free and $c(I) = R$.*

Actually, (i)$\Rightarrow$(ii)$\Rightarrow$(iii) is valid for arbitrary R, while only (iii)$\Rightarrow$(i) requires that R be quasi-local integrally closed. ((ii)$\Rightarrow$(i) can also be proved for slightly more general R, namely if R is the integral closure

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AMS 1970 *subject classifications*. Primary 13A15, 13B25, 13C10; Secondary 13B20.

Key words and phrases. Polynomial ring, flat module, torsion-free module, projective module, content, invertible ideal, finitely generated ideal, principal ideal.

[1] Research supported in part by NSF Grant GP-8972.

of a quasi-local ring R_0 and I is the extension of an ideal of $R_0[X]$, or if R is quasi-local and $J(\bar{R}) \subset R$.)

The following is a corollary to 1.1.

1.3 COROLLARY. *$R[X]/I$ is R-flat if and only if I is an invertible ideal and $c(I)=R$.*

In the case of a quasi-local R, the following theorem gives some additional interpretations of 1.2(i). We state the theorem for arbitrary R, in which case this condition translates to I being locally principal at primes of R and $c(I)=R$.

1.4 THEOREM. *Let ξ denote the equivalence class of X in $R[X]/I$. Then the following are equivalent:*

(i) *$R+\xi R+ \cdots +\xi^t R$ is flat for all $t \geqq 0$.*

(ii) *$R+\xi R+ \cdots +\xi^t R$ is flat for some $t \geqq 0$ for which* $1, \xi, \cdots, \xi^+$ *are linearly dependent over R.*

(iii) *For every prime ideal P of R, $IR_P[X]$ is principal generated by an element of* min $IR_P[X]$, *and $c(I)=R$.*

(iv) $I=(f_1, \cdots, f_n)$, $f_i \in$ min I, *and* $c(I)=R$.

(v) $c($min $I)=R$.

Moreover, if ξ is a regular element of $R[\xi]$, then (i)–(v) *are equivalent to*

(vi) *$R[\xi]$ and $R[1/\xi]$ are R-flat.*

Finally, if R is integrally closed, then (i)–(v) *are also equivalent to the equivalent assertions of* 1.3.

Recall that we have assumed throughout §1 that min I contains a regular element. Most of the results of this section are false without this assumption. A pertinent example is easily constructed as follows: Let A be any nonfinitely generated ideal of an absolutely flat ring R. (For example, take R to be the countable direct product of a field k and A to be the countable direct sum of copies of k.) Then let $I=AR[X]+XR[X]$. I is not finitely generated because A is not, and $R[X]/I$ is R-flat because R is absolutely flat.

2. **A theorem of Vasconcelos.** We now drop the assumption that min I contains a regular element, so that henceforth I is an arbitrary ideal of $R[X]$. The proofs of the above theorems, when I is finitely generated or principal, involve significant difficulties. If one is willing to skirt these difficulties, however, by means of various finiteness assumptions on either I or $R[X]/I$, then a number of further results can be obtained without great effort.

Vasconcelos has proved the following interesting theorem [**3**, p. 105].

Suppose R is noetherian. Then I is a projective ideal of $R[X]$ with $c(I)$ generated by an idempotent if and only if $R[X]/I$ is R-flat. Moreover, if $c(I)=R$ and $R[X]/I$ is R-projective, then $R[X]/I$ is a finite R-module.

2.1, 2.2, and 2.3 constitute a nonnoetherian generalization of this theorem. First, let us call an ideal A of R *locally trivial* if for every prime ideal P of R, either $AR_P=0$ or $AR_P=R_P$.

2.1 PROPOSITION. *If I is locally principal at primes of $R[X]$ and $c(I)$ is locally trivial, then $R[X]/I$ is R-flat. Conversely, if $R[X]/I$ is R-flat, then $c(I)$ is locally trivial and I is locally principal at any prime P' of $R[X]$ for which $IR[X]_{P'}$ is finitely generated.*

The first assertion of 2.1 is even true for a polynomial ring in arbitrarily many indeterminates.

2.2 THEOREM. *The following are equivalent:*

(i) *I is a projective ideal of $R[X]$, and $c(I)$ is generated by an idempotent.*

(ii) *$R[X]/I$ is R-flat, and I is a finitely generated ideal of $R[X]$.*

(iii) *I is a finitely generated flat ideal of $R[X]$, and $c(I)$ is generated by an idempotent.*

2.3 THEOREM. *Let R be a ring for which finitely generated flat R-modules are projective (e.g. a domain, a noetherian ring, or a quasi-local ring). If $R[X]/I$ is R-projective and $c(I)=R$, then $R[X]/I$ is a finite R-module and I is a finitely generated ideal.*

The conclusion of 2.3 can be sharpened in the case that R is quasilocal to read that I is a principal ideal of $R[X]$ generated by a regular element. Moreover, if R is an arbitrary ring and $R[X]/I$ is assumed *finite* and projective, then I is finitely generated; and, in fact, this remains valid when $R[X]$ is replaced by any finitely generated R-algebra.

We next give a new characterization of rings for which finitely generated flat modules are projective which gives an interesting perspective to our theorems of §1.

2.4 THEOREM. *The following statements for a ring R are equivalent:*

(i) *Finitely generated flat R-modules are projective.*

(ii) *For any ideal I of $R[X]$, $R[X]/I$ is a* **finite** *flat R-module implies I is a finitely generated ideal.*

Of course, the finiteness assumption in 2.4(ii) is the rub. Question. Does 2.4 remain valid when the word "finite" is deleted from 2.4(ii)?

It follows from 1.1 that any domain satisfies this modified 2.4(ii), so probably the next rings for which one should attempt to establish it are quasi-local rings.

3. **Nagata's theorem.** The theorem that originally motivated our work on these questions is the following theorem of Nagata [1, Theorem 3, p. 164]: If R is a valuation ring, then $R[X_1, \cdots, X_n]/I$ is R-flat implies I is a finitely generated ideal. It can be seen that this theorem is rather trivial in the case of one indeterminate. At present the only further progress on the case of a polynomial ring in n indeterminates seems to be the paper [2] of Nagata.

An analysis of Nagata's proof of this theorem shows that it actually yields the following result. Let $\varphi: R[X_1, \cdots, X_n] \to R[X_1, \cdots, X_n]/I$ be the canonical homomorphism, and let M_i denote the R-submodule of $R[X_1, \cdots, X_n]$ generated by the monomials of degree i.

3.1 THEOREM. *Suppose $R/J(R)$ is noetherian. If $\sum_{i=0}^{k} \varphi(M_i)$ is R-flat for every $k \geqq 0$, then I is a finitely generated ideal.*

3.2 COROLLARY. *Suppose $R/J(R)$ is noetherian. If I is a homogeneous ideal of $R[X_1, \cdots, X_n]$, or if R is semihereditary, then $R[X_1, \cdots, X_n]/I$ is R-flat implies I is a finitely generated ideal.*

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