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AN ALGEBRA ASSOCIATED TO A COMBINATORIAL GEOMETRY

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1. **Preliminaries.** A functor from a category of combinatorial geometries, or equivalently a category of geometric lattices, to a category of commutative algebras will be described, and some properties of this functor will be investigated. In particular, a cohomology will be associated to each point of a geometry and will be derived from the associated algebra.

If (G, S) is a geometry on a set S [1, p. 2.4], then $L(G)$, or simply L when no opportunity for confusion exists, denotes the associated geometric lattice of closed subsets of S. The rank function of L or G is denoted r.

A morphism

$$\sigma\colon (G, S) \to (G', S')$$

of geometries is a function

$$\sigma\colon S \cup \{0\} \to S' \cup \{0\}$$

such that $\sigma(0) = 0$ and the inverse image of a closed subset of S' is a closed subset of S. It is precisely the latter condition which is necessary and sufficient [1, p. 9.17] to extend σ to a strong map [1, p. 9.3]

$$\sigma\colon L(G) \to L(G').$$

The category of geometries with morphisms defined as above is equivalent to the category of geometric lattices and strong maps. These two categories will be used interchangeably throughout.

2. **The functor** $(G, S) \to A(G)$. Let k be a commutative ring (with 1). For any geometry (G, S), let $P(G)$ be the symmetric k-algebra on the free k-module on S; that is, the k-algebra of polynomials in S. Let $J(G)$ be the ideal of $P(G)$ generated by all monomials in S and differences of monomials in S of the forms

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$$x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n} \qquad \text{where } r(\bigvee x_i) < \nu_1+\nu_2+\cdots+\nu_n,$$
$$x_1x_2\cdots x_n - y_1y_2\cdots y_n \quad \text{where } \bigvee x_i = \bigvee y_i,$$

respectively and where n is any positive integer. Let $A(G) = P(G)/J(G)$. $J(G)$ is a homogeneous ideal of the graded algebra $P(G)$, and so $A(G)$ is a graded algebra. For each nonnegative integer n, let $A_n(G)$ denote the homogeneous component of $A(G)$ of degree n. That $A_n(G)=0$, for $n>r(G)$, follows from the definition of $J(G)$. The image of S in $A(G)$ generates $A(G)$ as an algebra, and it is easily seen that the natural algebra homomorphism of $P(G)$ onto $A(G)$ embeds S in $A(G)$ as a linearly independent subset. For this reason, identical symbols will be used to denote an element of S and the image of that element in $A(G)$ and $P(G)$.

For any nonnegative integer n such that $n \leqq r(L)$, let L_n denote the set of elements of L of rank n. For each $x \in L_n$, choose $x_1, x_2, \cdots, x_n \in S$ such that $x = \bigvee x_i$. The following proposition can easily be proved.

PROPOSITION 1. $\{\prod x_i \mid x \in L_n\}$ *is a basis for* $A_n(G)$.

Hence, $A(G)$ has a basis, and $A(G)$ is finite dimensional if and only if S is finite.

For $0 \leqq n \leqq r(L)$, let $B_n(G)$ denote the free k-module on L_n. Let $B(G) = \bigoplus B_n(G)$. For $x \in L_n$, $y \in L_m$ with $n, m \leqq r(L)$, the map

$$(x, y) \mapsto x \vee y, \quad \text{if } r(x \vee y) = n + m,$$
$$(x, y) \mapsto 0, \quad \text{if } r(x \vee y) < n + m,$$

determines a graded k-algebra structure on $B(G)$.

COROLLARY 1. $A(G) \simeq B(G)$ *as graded* k*-algebras.*

Let $\sigma: (G, S) \to (G', S')$ be a morphism of geometries. For $x \in S$, the association

$$x \mapsto \sigma(x) \in A(G'), \quad \text{if } \sigma(x) \in S',$$
$$x \mapsto 0 \in A(G'), \quad \text{if } \sigma(x) = 0,$$

uniquely determines an algebra homomorphism of $P(G)$ into $A(G')$. Since σ, considered as a strong map of $L(G)$ to $L(G')$, is supremum-preserving and satisfies $r(\sigma(w)) \leqq r(w)$ for any $w \in L(G)$, the kernel of this algebra homomorphism contains $J(G)$. So there is uniquely determined a homomorphism

$$A(\sigma): A(G) \to A(G')$$

of algebras. In fact, $A(\sigma)$ is a homomorphism of graded algebras. It follows easily that

PROPOSITION 2. *The association $\sigma \mapsto A(\sigma)$ is a functor from the category of geometries, or equivalently the category of geometric lattices and strong maps, to the category of commutative k-algebras.*

PROPOSITION 3. *If $(G \oplus G', S \cup S')$ is the coproduct (direct sum) in the category of geometries of (G, S) and (G', S'), then $A(G \oplus G') \simeq A(G) \otimes A(G')$. That is, A preserves coproducts.*

PROOF. $L(G \oplus G') \simeq L(G) \times L(G')$ where the lattice structure on the cartesian product is defined componentwise. The linear map of $B(G \oplus G')$ to $B(G) \otimes B(G')$ determined by $(x, x') \mapsto x \otimes x'$, where $x \in L(G)$ and $x' \in L(G')$, is easily seen to be an algebra isomorphism, and the proposition follows from Corollary 1.

Notice that $A(G) \otimes A(G')$ is the tensor product of commutative algebras and is not the tensor product of graded algebras. It is for this reason that A is considered a functor into the category of commutative algebras rather than into the category of graded algebras.

Let e be a closed subset of S where (G, S) is a geometry. So $e \in L(G)$. If $r(e) = n > 0$, choose $e_1, e_2, \cdots, e_n \in S$ such that $e = \bigvee e_i$. Let $\operatorname{ann}(e)$ denote the annihilator of $\prod e_i$ in $A(G)$. That is,

$$\operatorname{ann}(e) = \{y \in A(G) \mid y(\prod e_i) = 0\}.$$

$\operatorname{ann}(e)$ is a homogeneous ideal of $A(G)$. Let (G_e, e) denote the subgeometry [1, p. 4.1] of (G, S) on the closed subset e. Let $A(G)_e$ denote the subalgebra of $A(G)$ generated by the subset e of S. Let $(G/e, S_e)$ denote the geometry associated to the contraction [1, p. 4.1] of (G, S) to the open subset $S - e$ (the contraction is, in general, only a pregeometry).

PROPOSITION 4. $A(G_e) \simeq A(G)_e$.

PROOF. The image of the linear map of $P(G_e)$ into $A(G)$ induced by $\prod x_i^{y_i} \mapsto \prod x_i^{y_i}$, where each $x_i \in e$, is $A(G)_e$, and its kernel is precisely $J(G_e)$.

PROPOSITION 5. $A(G/e) \simeq A(G)/\operatorname{ann}(e)$.

PROOF. The geometric lattice $L(G/e)$ is isomorphic to the interval $[e, 1]$ $(1 = S)$ of $L(G)$ [1, p. 4.2]. The map $\sigma: L(G) \to [e, 1]$ defined by $\sigma(x) = x \vee e$ is a strong map. The induced surjection $A(\sigma): A(G) \to A(G/e)$ given by $x \mapsto x(\prod e_i)$, $x \in A(G)$, has kernel $\operatorname{ann}(e)$.

Propositions 4 and 5 also hold when the closed subset e is replaced by an arbitrary subset.

3. **Cohomology relative to a point of** (G, S). Let $p \in S$ where (G, S) is a geometry. Define $\partial(p) \in \mathrm{End}_k(A(G))$ by $\partial(p)(x) = px$, $x \in A(G)$. $\partial(p)$ is an endomorphism of degree 1, and $\partial(p)^2 = 0$ since $p^2 = 0$ in $A(G)$. Letting $m = r(G)$, there is induced a cochain complex

$$k = A_0(G) \xrightarrow{\partial(p)} A_1(G) \xrightarrow{\partial(p)} \cdots \xrightarrow{\partial(p)} A_{m-1}(G) \xrightarrow{\partial(p)} A_m(G).$$

Let $H^n(G, p) = Z^n(G, p)/B^n(G, p)$ where, for $0 < n < m$, $Z^n(G, p)$ is the kernel of $\partial(p): A_n(G) \to A_{n+1}(G)$ and $B^n(G, p)$ is the image of $\partial(p): A_{n-1}(G) \to A_n(G)$. Clearly, $Z^n(G, p) = \mathrm{ann}_n(p) =$ homogeneous component of degree n of $\mathrm{ann}(p)$. Recall that, for $0 \leqq n \leqq m$, L_n denotes the set of elements of $L = L(G)$ of rank n and that, for $x \in L_n$, $x_1, x_2, \cdots, x_n \in S$ were chosen so that $x = \bigvee x_i$. Let $L_n(p) = \{x \in L_n \mid p \leqq x\}$. It is easily seen that $\mathrm{ann}_n(p)$, $0 < n < m$, has a basis consisting of the set

$$\{\prod x_i \mid x \in L_n(p)\}$$

and, for each $w \in L_{n+1}(p)$, the set

$$\{\prod x_i - \prod y_i \mid y \in L_n,\ x \neq y,\ y \vee p = w\}$$

where $x \in L_n$ is fixed and $x \vee p = w$. Let $|T|$ denote the cardinality of a set T. The following proposition is easily proved.

PROPOSITION 6. *Let (G, S) be a geometry with $|S| < \infty$. For any $p \in S$ and $0 < n < r(G)$, $H^n(G, p)$ is a free k-module of dimension $|L_n| - |L_{n+1}(p)| - |L_n(p)|$.*

For example, if (G, S) is the free geometry on a finite set S ($L(G)$ = Boolean algebra of all subsets of S), then $H^n(G, p) = 0$ for all $p \in S$ and $0 < n < |S|$.

For a nontrivial example, let E be the set of edges of a finite graph with no loops. Let (G, E) be the bond closure geometry of the graph. That is, if $T \subseteq E$, then the G-closure of T consists of all edges both of whose endpoints belong to the same connected component of T. Let $e \in E$. Then the dimension of $H^1(G, e)$ is the number of 3-circuits (triangles) containing e. If the graph is connected and m is the number of edges in a maximal tree (m is the rank of the geometry), then the dimension of $H^{m-1}(G, e)$ is one less than the number of bonds in the graph containing e.

Note also that if $\sigma: (G, S) \to (G', S')$ is a morphism of geometries and if $\sigma(p) \neq 0$, then $A(\sigma): A(G) \to A(G')$ induces a homomorphism $A(\sigma): H^n(G, p) \to H^n(G', \sigma(p))$ for each $0 < n < r(G)$ where $H^n(G', \sigma(p)) = 0$ for $n \geqq r(G')$.

4. **Remarks.** The functor $(G, S) \to A(G)$ could have been considered as a functor from the category of pregeometries in which case the algebra attached to a pregeometry would be isomorphic to the algebra attached to the associated geometry. The construction of $A(G)$ can be adapted to the definition of geometry in [2] at the expense of clarity of notation.

Can the algebras $A(G)$ be abstractly characterized? Can a characterization be given of the ideals of $A(G)$ arising as kernels of $A(\sigma) : A(G) \to A(G')$ where $\sigma : (G, S) \to (G', S')$ is a map of geometries? Can (G, S) be recovered from $A(G)$? For which (G, S) is $H^n(G, p)$ independent of $p \in S$?

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