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A NONSMOOTHABLE KNOT

BY RICHARD K. LASHOF

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In this note we prove the existence of a locally flat topological embedding of S^3 in S^5, which is not equivalent to a smooth embedding under homeomorphism of S^5.

The proof depends heavily on the results of Cappell and Shaneson [CS]. In their paper they describe a homotopy equivalence h of $M=S^3\times S^1\#r(S^2\times S^2)$ onto itself, which is not homotopic to a diffeomorphism (Proposition 3.2 and Example 1.6). It is not known whether h can be taken to be a homeomorphism; however $h\times 1:M\times S^1\to M\times S^1$ is homotopic to a homeomorphism but not a diffeomorphism. Thus the homeomorphism, say $k:M\times S^1\to M\times S^1$, represents the nontrivial element $\alpha\in\pi_3(\mathrm{Top}/\mathrm{PL})=Z_2$, i.e., the obstruction in $H^3(M\times S^1;\pi_3(\mathrm{Top}/\mathrm{PL}))$ to smoothing k, (see [KS] and Remark 2).

Using the trivial normal bundle of $S^3\subset S^3\times S^1\#r(S^2\times S^2)$, we have a smooth embedding of $S^3\times R$ in M and $S^3\times R^2$ in $M\times R$. Now k has a covering map $\hat{k}:M\times R\to M\times R$, also representing α, since the obstruction depends only on k in a neighborhood of S^3. We will define a smooth embedding $i:M\times R\to S^5$; then $f=i\circ\hat{k}|S^3\times R^2:S^3\times R^2\to S^5$ will also represent α, and we will show that $f|S^3\times 0$ is the desired locally flat embedding.

To define i, take a standard embedding of S^3 in S^5, and then its normal sphere bundle $S^3\times S^1$ will be smoothly embedded in S^5 with a trivial normal bundle. We may add trivial 2-handles to $S^3\times S^1\times I\subset S^5$, to embed M (in fact, the cobordism from $S^3\times S^1$ to M) in S^5 with a trivial normal bundle. Thus we get our smooth embedding $i:M\times R\to S^5$.

Now suppose there exists a homeomorphism $g:S^5\to S^5$ such that $g\circ f|S^3\times 0$ is smooth. By the uniqueness of codimension two normal bundles [K], we may assume $g\circ f:S^3\times R^2\to S^5$ is smooth.[1] On the other hand, any homeomorphism of S^5 is isotopic to a diffeomorphism [KS]; say g is isotopic to the diffeomorphism d. Then $d^{-1}\circ g\circ f$ is

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[1] Reference to [K] may be avoided as follows: $g\circ f(S^3\times R^2)$ is diffeomorphic to $S^3\times R^2$ by engulfing, and hence is the trivial smoothing. Therefore, $g\circ f$ is ambient isotopic to a smooth embedding; and thus we may assume $g\circ f$ is smooth.

smooth, but $d^{-1} \circ g$ is isotopic to the identity. Therefore f is isotopic to a smooth embedding, and represents the trivial class in $\pi_3(\mathrm{Top}/\mathrm{PL})$. Contradiction. Thus we have proved:

THEOREM. *There is a locally flat topological embedding* $f: S^3 \to S^5$ *which is not equivalent to a smooth embedding; i.e., there exists no homeomorphism* g *of* S^5 *such that* $g \circ f$ *is smooth.*

REMARKS. 1. $S^5 - f(S^3)$ cannot be the homotopy type of S^1, by Stallings' unknotting theorem [S].

2. Of course, f is not equivalent to a locally flat piecewise linear embedding either; but since $\pi_3(\mathrm{Top}/\mathrm{PL}) = \pi_3(\mathrm{Top}/0)$, the obstruction is the same for smoothing [L].

3. The smoothing $(S^3 \times R^2)_\alpha$ induced by $f: S^3 \times R^2 \to S^5$ is not diffeomorphic to $S^3 \times R^2$ by results of Kirby and Siebenmann. On the other hand, by taking a smooth embedding of S^1 in $S^5 - f(S^3 \times 0)$, representing a generator of $H_1(S^5 - f(S^3 \times 0))$, we have a smooth embedding $f: (S^3 \times R^2)_\alpha \to S^5$-normal tube of $S^1 = (S^3 \times R^2)_0$; where $(S^3 \times R^2)_0$ is the standard smoothing.

4. Since α is of order 2, the embedding $f: S^3 \times R^2 \to S^3 \times R^2$ given in Remark 3 induces $(S^3 \times R^2)_0$ from $(S^3 \times R^2)_\alpha$; i.e., there is a smooth embedding of S^3 in $(S^3 \times R^2)_\alpha$ which is a homotopy equivalence.

It follows that there is also a smooth embedding of S^3 in $(S^3 \times T^2)_\alpha$, T^2 the 2-torus, which represents a generator of $\pi_3(S^3 \times T^2)$.

5. There appears to be as many nonsmoothable knots as smoothable ones, since we can take any smooth embedding of S^3 in S^5 to define i.

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UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637