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MORE DISTANT THAN THE ANTIPODES

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1. **Introduction.** Let X be a real normed linear space, and let $\Sigma(X)$ be its unit ball, with the boundary $\partial\Sigma(X)$. If $\dim X \geqq 2$, δ_X denotes the inner metric of $\partial\Sigma(X)$ induced by the norm (cf. [**1**, §3]). If no confusion is likely, we write Σ, $\partial\Sigma$, δ. In [**1**] we introduced and discussed parameters of X based on the metric structure of $\partial\Sigma$; among them are $D(X) = \sup\{\delta(p, q) : p, q \in \partial\Sigma\}$, the *inner diameter* of $\partial\Sigma$, and $M(X) = \sup\{\delta(-p, p) : p \in \partial\Sigma\}$, half the *perimeter* of Σ. Obviously, $M(X) \leqq D(X)$, and it was conjectured [**1**, Conjecture 9.1] that $M(X) = D(X)$ in every case, i.e., that "no pair of points of $\partial\Sigma$ is more distant in $\partial\Sigma$ than the most distant antipodes." This equality was shown to hold if $\dim X = 2$ or $\dim X = 3$ [**1**, Theorems 5.4, 5.8], if $D(X) = 4$ [**3**], if X is an L-space [**4**].

In this paper we explode this conjecture by showing that $M(X) = 2$, $D(X) = 3$ for $X = C_0((0, 1])$, the space of continuous real-valued functions on $(0, 1]$ that tend to 0 at 0, with the supremum norm. We observe that this failure of the conjecture is "as strong as possible," since $2D(X) \leqq M(X) + 4$ for every normed space X [**3**, Theorem 1]. The present result is a simple specific instance of the evaluation of $M(X)$, $D(X)$ for many spaces of continuous functions, which will be carried out in a forthcoming paper. It has appeared useful, however, to give a separate account of this very simple example. In addition, Lemma 1 is required for the general theory. The conjecture remains unresolved, and interesting, for spaces of finite dimension greater than three.

We shall use the terminology, notations, and elementary results of §§1–3 of [**1**]. In particular, a *subspace* of X is a linear manifold in X, not necessarily closed, provided with the norm of X. If Y is a subspace of X, we obviously have

$$(1) \qquad \delta_Y(p, q) \geqq \delta(p, q), \qquad p, q \in \partial\Sigma(Y).$$

Instead of dealing with the space $C_0((0, 1])$, we prefer, for technical reasons, to consider the space $C_\pi([-1, 1])$ of odd continuous real-valued functions on $[-1, 1]$ with the supremum norm. The two

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spaces are obviously congruent, and therefore any metrical property of one implies the same metrical property of the other. *In the rest of this paper, X shall always stand for $C_\pi([-1, 1])$.*

2. **The perimeter.** We consider the special function $u \in X$ defined by $u(t) = t$, $t \in [-1, 1]$.

LEMMA 1. $\delta(-u, u) = 2$.

PROOF. For each given integer $n > 1$, let $R_n = l^\infty(\{1, \cdots, n\})$ be the Banach space of sequences of length n of real numbers, with the maximum norm. The proof will depend on the computation of the length of certain polygonal curves in $\partial\Sigma(R_n)$, carried out in [**2**].

Let Y_n be the closed subspace of X consisting of the piecewise linear odd continuous real-valued functions on $[-1, 1]$ with "corners" at most at $\pm(2k-1)(2n-1)^{-1}$, $k = 1, \cdots, n$. Define the linear mapping $\Phi_n: Y_n \to R_n$ by $(\Phi_n f)(j) = f((2n-4j+3)(2n-1)^{-1})$, $j = 1, \cdots, n$. Since the mapping $j \mapsto 2n-4j+3: \{1, \cdots, n\} \to \{\pm(2k-1): k = 1, \cdots, n\}$ is injective and the image contains exactly one of each pair of opposites, Φ_n is bijective; since a piecewise linear function attains its extrema at "corners," Φ_n is isometric. Hence Φ_n is a congruence.

Now $u \in Y_n$; we consider $\Phi_n u \in \partial\Sigma(R_n)$ and compute

$$(2) \qquad (\Phi_n u)(j) = (2n - 4j + 3)(2n - 1)^{-1}, \qquad j = 1, \cdots, n.$$

On the other hand, we consider $p_0 \in \partial\Sigma(R_n)$ given by

$$(3) \qquad p_0(j) = (n - 2j + 1)(n - 1)^{-1}, \qquad j = 1, \cdots, n;$$

we know from [**2**, Lemma 4] that

$$(4) \qquad \delta_{R_n}(-p_0, p_0) \leqq 2n(n - 1)^{-1}$$

(in fact, equality holds). Now $(\Phi_n u)(1) = p_0(1) = 1$, so the straight-line segment with endpoints $\Phi_n u$, p_0 lies entirely in $\partial\Sigma(R_n)$; therefore, from (2), (3),

$$(5) \qquad \begin{aligned} \delta(p_0, \Phi_n u) &= \|\Phi_n u - p_0\| \\ &= 2(2n - 1)^{-1}(n - 1)^{-1} \max\{j - 1 : j = 1, \cdots, n\} \\ &= 2(2n - 1)^{-1}. \end{aligned}$$

Since $\Phi_n: Y_n \to R_n$ is a congruence, (1), (4), (5) yield

$$\begin{aligned} 2 &= \|u - (-u)\| \leqq \delta(-u, u) \leqq \delta_{Y_n}(-u, u) = \delta_{R_n}(-\Phi_n u, \Phi_n u) \\ &\leqq \delta_{R_n}(-\Phi_n u, -p_0) + \delta_{R_n}(-p_0, p_0) + \delta_{R_n}(p_0, \Phi_n u) \\ &\leqq 4(2n - 1)^{-1} + 2n(n - 1)^{-1} = 2 + 2(4n - 3)(n - 1)^{-1}(2n - 1)^{-1}. \end{aligned}$$

The integer n was arbitrarily great; we conclude that $\delta(u, -u)=2$.

THEOREM 2. *For every* $f\in\partial\Sigma$, $\delta(-f, f)=2$. *Consequently*, $M(X)=2$.

PROOF. Since $[-1, 1]$ is connected and f is odd, we have $f([-1, 1]) = [-1, 1]$. Since the composition of odd functions is odd, we conclude that the linear mapping $g\mapsto g\circ f: X\to X$ is isometric, hence a congruence of X onto a subspace Y of X. Now $(\pm u)\circ f=\pm f\in Y$; by Lemma 1 and (1) we therefore have

$$2 \leqq \delta(-f, f) \leqq \delta_Y(-f, f) = \delta_Y(-u\circ f, u\circ f) = \delta(-u, u) = 2.$$

3. The inner diameter.

LEMMA 3. *Define* $v, w\in\partial\Sigma$ *by*

$$v(t) = -v(-t) = t - \tfrac{1}{2} + \left| t - \tfrac{1}{2}\right|, \qquad 0 \leqq t \leqq 1,$$
$$w(t) = -w(-t) = -t - \tfrac{1}{2} + \left| t - \tfrac{1}{2}\right|, \qquad 0 \leqq t \leqq 1.$$

Then $\delta(v, w)\geqq 3$.

PROOF. Let $\mathfrak{c}$ be any curve from v to w in $\partial\Sigma$, and r a given number, $0\leqq r<1$. Since $\|v-v\|=0$, $\|v-w\|=2$, there exists a point z on $\mathfrak{c}$ such that $\|z-v\|=r$. Since $z\in\partial\Sigma$ there exists $t\in[-1, 1]$ such that $z(t)=1$. Now $v(t)\geqq z(t)-\|z-v\|=1-r>0$. From the definition of v and w we have $t>\frac{1}{2}$, whence $w(t)=-1$. Then

$$l(\mathfrak{c}) \geqq \|w - z\| + \|z - v\| \geqq |w(t) - z(t)| + r = 2 + r.$$

Since r was arbitrarily close to 1, we have $l(\mathfrak{c})\geqq 3$. Since $\mathfrak{c}$ was an arbitrary curve from v to w in $\partial\Sigma$, we indeed have $\delta(v, w)\geqq 3$.

REMARK. It is easy to show directly that $\delta(v, w)=3$; there exists, in fact, a curve from v to w in $\partial\Sigma$ consisting of two straight line segments end-to-end, of respective lengths 1 and 2: the intermediate endpoint is $z\in\partial\Sigma$ defined by

$$z(t) = -z(-t) = t - \tfrac{3}{2} + 3\left| t - \tfrac{1}{2}\right|, \qquad 0 \leqq t \leqq 1.$$

The verification is left to the reader.

THEOREM 4. $D(X)=3$.

PROOF. By [3, Theorem 1], $2D(X)\leqq M(X)+4$; since $M(X)=2$ by Theorem 2, we conclude, using Lemma 3, that

$$3 \leqq \delta(v, w) \leqq D(X) \leqq \tfrac{1}{2}(2 + 4) = 3,$$

so that equality holds.

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