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CONVERSE THEOREMS AND EXTENSIONS IN CHEBYSHEV RATIONAL APPROXIMATION TO CERTAIN ENTIRE FUNCTIONS IN $[0, +\infty)$

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Recently, it has been shown that the problem of rational approximation to e^{-x} in $[0, +\infty)$ arises naturally in numerical methods for approximating solutions of heat-conduction-type partial differential equations [1], [2]. This special approximation problem leads one to the general question of approximating functions on the half line $[0, +\infty)$. In this paper we wish to announce two results of this study which are in the spirit of work done by S. N. Bernstein. A complete description of this work with proofs of these results and additional results will appear elsewhere.

In order to state these results, we need the following notation. For any nonnegative integer m, let π_m denote the collection of all real polynomials of degree at most m. For given $r>0$ and $s>1$, let $\mathcal{E}(r, s)$ denote the unique open ellipse in the complex plane with foci at $x=0$ and $x=r$ and semimajor and semiminor axes a and b such that $b/a=(s^2-1)/(s^2+1)$. Finally, if $f(z)$ is any entire function, we set

$$\tilde{M}_f(r, s) = \sup\{ |f(z)| : z \in \mathcal{E}(r, s)\}$$

and

$$M_f(r) = \sup\{ |f(z)| : |z| = r\}.$$

We now state our main results.

THEOREM 1. *Let $f(x)$ be a real continuous function on $[0, +\infty)$ with at most a finite number of points $\{x_i\}_{i=1}^m$ in $[0, +\infty)$ for which $f(x_i)=0$. Assume that there exist a sequence of real polynomials $\{p_n(x)\}_{n=0}^\infty$, with $p_n \in \pi_n$ for each $n \geqq 0$, and a real number $q>1$ such that*

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$$\limsup_{n\to\infty}\left\{\left\|\frac{1}{f(x)}-\frac{1}{p_n(x)}\right\|_{L_\infty[0,\infty)}\right\}^{1/n}=\frac{1}{q}<1.$$

Then, there exists an entire function $F(z)$ with $F(x)=f(x)$ for all $x\geqq 0$, and F is of finite order ρ, i.e.,

$$\limsup_{r\to\infty}\frac{\ln\ln M_F(r)}{\ln r}\equiv\rho<\infty.$$

In addition, for each $s>1$, there exist real numbers $K=K(q,s)>0$, $\theta=\theta(q,s)>1$ and $r_0=r_0(q,s)>0$ such that $\tilde{M}_F(r,s)\leqq K(\|f\|_{L_\infty[0,r]})^\theta$ for all $r\geqq r_0$. If, for each $s>1$, $\tilde{\theta}(s)$ is defined by

$$\limsup_{r\to\infty}\left\{\frac{\ln\tilde{M}_F(r,s)}{\ln\|f\|_{L_\infty[0,r]}}\right\}=\tilde{\theta}(s)$$

when $\|f\|_{L_\infty[0,r]}$ is unbounded as $r\to\infty$, and $\tilde{\theta}(s)\equiv 1$ otherwise, then the order ρ of F satisfies

$$\rho\leqq\inf_{s>1}\left\{\frac{\ln\tilde{\theta}(s)}{\ln[\frac{1}{2}+\frac{1}{4}(s+1/s)]}\right\},$$

and this upper bound for the order ρ is in general best possible.

THEOREM 2. *Let $f(z)=\sum_{k=0}^\infty a_k z^k$ be an entire function with nonnegative Taylor coefficients and $a_0>0$. If there exist real numbers $s>1$, $A>0$, $\theta>0$ and $r_0>0$ such that $\tilde{M}_f(r,s)\leqq A(\|f\|_{L_\infty[0,r]})^\theta$ for all $r\geqq r_0$, then there exist a sequence of real polynomials $\{p_n(x)\}_{n=0}^\infty$ with $p_n\in\pi_n$ for each $n\geqq 0$, and a real number $q\geqq s^{1/1+\theta}>1$ such that*

$$\limsup_{n\to\infty}\left\{\left\|\frac{1}{f(x)}-\frac{1}{p_n(x)}\right\|_{L_\infty[0,\infty)}\right\}^{1/n}=\frac{1}{q}<1.$$

The requirement that the Taylor coefficients be nonnegative in Theorem 2 is due to a technical difficulty in the proof of the theorem. Examples can be given to show that this requirement is not necessary for geometric convergence to occur.

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