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THE CLASSIFICATION OF FREE ACTIONS OF CYCLIC GROUPS OF ODD ORDER ON HOMOTOPY SPHERES

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Consider the cyclic group Z_p acting freely on a homotopy sphere Σ^{2n+1}. This action is a map $\mu: Z_p \times \Sigma \to \Sigma$. We shall consider the three cases where the action is smooth, piecewise linear (PL) or topological. If we pick a generator $T \in Z_p$ then we may say that two actions $\mu_i: Z_p \times \Sigma_i^{2n+1} \to \Sigma_i^{2n+1}$, $i=1, 2$, are equivalent actions if there is an equivalence $f: \Sigma_1 \to \Sigma_2$ in the appropriate category (i.e. f is diffeomorphism, PL equivalence or homeomorphism depending on whether μ_i are smooth, PL or topological) such that $f\mu_1(T, x) = \mu_2(T, f(x))$. Among smooth actions we have the linear actions on $S^{2n+1} \subset C^{n+1}$ where the action is given by

$$(z_0, \cdots, z_n) \to (\exp[2\pi i\theta_0/p]z_0, \cdots, \exp[2\pi i\theta_n/p]z_n)$$

where $\theta_0, \cdots, \theta_n$ are integers between 1 and $p-1$, prime to p. The quotient of S^{2n+1} by this action is the Lens space $L^{2n+1}(p; \theta_0, \cdots, \theta_n)$, and it is well known that for any free Z_p action on Σ^{2n+1}, the orbit space Σ/Z_p is homotopy equivalent to $L^{2n+1}(p; \theta_0, \cdots, \theta_n)$ for some appropriate choice of $\theta_0, \cdots, \theta_n$.

Now two Z_p-actions (μ_i, Σ_i), $i=1, 2$, are equivalent if and only if the orbit spaces Σ_i/Z_p are equivalent in the appropriate category. The equivalence is given by an orientation preserving isomorphism $g: \Sigma_1/Z_p \to \Sigma_2/Z_p$ such that $g_*(T) = T$ where T is identified with the corresponding element in $\pi_1(\Sigma_i/Z_p)$. Thus the problem of equivalence of Z_p actions is the problem of classifying manifolds of the homotopy type of Lens spaces up to equivalence in the corresponding category.

In this note we announce a procedure for doing this when p is odd, $n \geqq 2$. Some related work has been done by R. Lee [3], [4]. The result for $p=2$ was previously found by Lopez de Medrano [5] and Wall [11] and the answer for p odd is quite analogous. One uses the theory of surgery to study the problem, and calculates as far as possible the terms arising. As usual one arrives at a complete classification in the

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PL and topological categories but the smooth case must await more profound knowledge of the homotopy of spheres.

We recall [9] that a (simple) homotopy structure on a manifold X^n is a pair (M^n, f), M^n a manifold in the appropriate category, $f: M^n \to X^n$ a (simple) homotopy equivalence. Two structures (M_i, f_i), $i=0, 1$, on X are *concordant* if there is a structure (W^{n+1}, F) on $X \times [0, 1]$, $F: W \to X \times [0, 1]$ such that ∂W is isomorphic to $M_0 \cup M_1$ (in the category) and $F| M_i = f_i$ into $X \times i$, $i=0, 1$. The set of concordance classes of structures is denoted by $\mathcal{S}^\epsilon_{\mathcal{C}}(X)$ where $\epsilon = h$ or s (for homotopy or simple homotopy structures), $\mathcal{C} =$ Diff, PL or Top is the category of manifolds. It follows from the s-cobordism theorem that $(M_0, f_0) = (M_1, f_1)$ in $\mathcal{S}^s(X)$ if and only if $f_1^{-1} f_0$ is homotopic to an isomorphism, provided the dimension of $X = n \geqq 5$.

The endproduct of the theory of "Surgery on maps" is the following exact sequence:

$$\text{(S)} \quad \to L^\epsilon_{n+1}(\pi) \xrightarrow{\omega} \mathcal{S}^\epsilon(X) \xrightarrow{\eta} [X, G/H] \xrightarrow{\sigma} L_n(\pi) \quad \text{where } \epsilon = h \text{ or } s,$$

$H=0$, PL or Top depending on whether we are in the Diff, PL or Top category, $\mathcal{S}$ is structures in that category, $\pi = \pi_1(X)$, and $L^\epsilon_k(\pi)$ is a covariant functor on finitely presented groups into abelian groups and $L^\epsilon_{n+1}(\pi)$ acts on $\mathcal{S}^\epsilon(X)$ to make the sequence exact in the strong sense.

There is a natural relationship between this exact sequence and the representation theory of π. Starting from the complex representation ring $R(\pi)$ of π we form the "localized ring" $S^{-1}R(\pi)$, where $S \subset R(\pi)$ is the multiplicative set $\{\lambda^k | \lambda = N - R, k = 0, 1, 2 \cdots\}$, N is the order of π and R is the regular representation of π. There is a commutative diagram [8]

$$\text{(T)} \quad \begin{array}{ccccccc} \cdots \to L^\epsilon_{n+1}(\pi) & \xrightarrow{\omega} & \mathcal{S}^\epsilon(X) & \xrightarrow{\eta} & [X, G/H] & \xrightarrow{\sigma} & L^\epsilon_n(\pi) \\ \downarrow \chi & & \downarrow A & & & & \downarrow \psi \\ R(\pi) & \to & S^{-1}R(\pi) & \to & & & S^{-1}R(\pi)/R(\pi) \end{array}$$

when π is finite and n is odd.

We now restrict attention to the case $\pi = Z_p$ and X is the orbit space of a free Z_p action on a homotopy sphere. Then X is homotopy equivalent to a Lens space (but not necessarily simple homotopy equivalent to one). We now discuss the sequence (T) with $X = L^{2n+1}(p; \theta_0, \cdots, \theta_n)$, and $\pi = \pi_1(X) = Z_p$. It follows that elements of $\mathcal{S}^s(L^{2n+1}(p; \theta_0, \cdots, \theta_n))$ are in 1-1 correspondence with equivalence classes of Z_p actions whose orbit space is simple homotopy equivalent to $L^{2n+1}(p; \theta_0, \cdots, \theta_n)$ and all other actions may be obtained from these by action of the group $L_{2n}^h(Z_p)$ and the Whitehead group $\mathrm{Wh}(Z_p)$ (on the orbit space).

THEOREM 1 [2]. *For $X = L^{2n+1}(p; \theta_0, \cdots, \theta_n)$, p odd $n \geqq 2$, $\epsilon = s$ or h the map η is onto.*

THEOREM 2 ([7] AND [12]). *$L_{2k}^s(Z_p)_0$ is a free abelian group of rank $\frac{1}{2}(p-1)$ and acts freely on $\mathcal{S}^s(X)$ for any X with $\pi_1(X) = Z_p$ (in each category) $k \geqq 3$; moreover, the image of χ in $R(\pi)$ is of rank $\frac{1}{2}(p-1)$.*

Here $L_n^s(\pi)_0 = \text{kernel } (L_n^s(\pi) \to L_n^s(0))$.

The map A can be described in several different but related ways:

(a) As a signature invariant generalizing the Z_2-signature used in [11].

(b) As a signature invariant of the type defined by Atiyah-Singer as in [7].

(c) As an element of $KO(BZ_p)$ using the KO theory orientation of PL bundles defined and exploited by Sullivan [10].

Let $[M, f] \in \mathcal{S}^\epsilon(L^{2n+1}(p; \theta_0, \cdots, \theta_n))$ and if Σ is the total space of the Z_p bundle over M induced by f, then some multiple $p^k\Sigma$ bounds a manifold W supporting a free action of Z_p. Set $A[M, f] = (p-R)^{-k} \cdot \mathrm{Sgn}(Z_p, W)$ where $\mathrm{Sgn}(Z_p, W)$ is the representation of Z_p constructed from the Z_p invariant bilinear form on $\hat{H}^{n+1}(W, \mathbf{C})$ defined by the cup product pairing (see [1]).

In particular one may show that the definition of A does not depend on the choice of homotopy equivalence. Namely, if (M_i, f_i) represent elements $x_i \in \mathcal{S}(L^{2n+1}(p; \theta_0, \cdots, \theta_n))$, $i = 1, 2$, and if there is an orientation preserving PL (topological) equivalence $g: M_1 \to M_2$ such that $g_*(T) = T$ (in $\pi_1(M_i) = Z_p$) then $A(x_1) = A(x_2)$.

Theorems 1 and 2 together give one a classification theorem of sorts in each category. For the categories PL and Top, G/PL and G/Top are amenable to calculations. Namely from the work of Sullivan [10] in the PL case, which for odd primes the work of Kirby-Siebenmann shows is equivalent to the Top case, we have:

THEOREM 3. *If $X = L^{2n+1}(p; \theta_0, \cdots, \theta_n)$, p odd, then $[X, G/H] \cong \widetilde{KO}(X)$, where $H = \mathrm{PL}$ or Top.*

THEOREM 4. *The invariant A characterizes the PL or topological type of a simple homotopy structure on $L^{2n+1}(p; \theta_0, \cdots, \theta_n)$, $n \geqq 2$, i.e. if $(M_i, f_i) \in \mathcal{S}_{\mathrm{PL}}^s(L^{2n+1}(p; \theta_0, \cdots, \theta_n))$, $n \geqq 2$, $i = 0, 1$, then $(M_0, f_0) = (M_1, f_1)$ if and only if $A(M_0, f_0) = A(M_1, f_1)$.*

Following Milnor [6, pp. 404–406] one can define a torsion invariant Δ for a manifold $M=\Sigma^{2n+1}/Z_p$, where $\Delta(M)\in Q(Z_p)/\Sigma$, the rational group ring modulo the ideal generated by the sum of the group elements. If $\Delta(M)=\Delta(M')$ then M and M' are simple homotopy equivalent. Furthermore:

THEOREM 5 [12]. *Two actions of Z_p, p odd, on homotopy $(2n+1)$-spheres, $n\geqq 2$, are* PL *or topologically equivalent if and only if their invariants A and Δ agree.*

One may also describe which pairs Δ and A arise as the invariants of such Z_p actions but we will not do this here.

Let $\mu:Z_p\times S^{2n+1}\to S^{2n+1}$ be a free PL or topological action on the sphere. We may define the *suspension* $\Sigma\mu$, of μ by taking the induced action on the join $S^{2n+1}*S^1$, using the action $T(e^{2i\pi t})=e^{2i\pi(t+(1/p))}$ on S^1. Then $\Sigma\mu$ is a free PL or topological action on S^{2n+3} and if the orbit space of $\mu=S^{2n+1}/Z_p$ is homotopy equivalent to $L^{2n+1}(p;\theta_0,\cdots,\theta_n)$ then the orbit space of μ_θ, S^{2n+3}/Z_p is homotopy equivalent to $L^{2n+3}(p;\theta_0,\cdots,\theta_n,1)$.

Using Theorem 5 and studying the effect of suspension on A and Δ one obtains:

COROLLARY 1 [12]. *If $n>2$, then every* PL *or topological action on S^{2n+1} is the suspension of a unique action on S^{2n-1}.*

Note that the suspension of a smooth action $\mu:Z_p\times\Sigma^{2n+1}\to\Sigma^{2n+1}$ will be smoothable if and only if the orbit space of the diagonal action of Z_p on $\Sigma^{2n+1}\times S^1$ is diffeomorphic to $S^{2n+1}\times S^1$, which is equivalent to the condition that Σ^{2n+1} is diffeomorphic to S^{2n+1}. In that case there may be many smoothings of it.

COROLLARY 2. *Let $\mu:Z_p\times\Sigma^{2n+1}\to\Sigma^{2n+1}$ be a smooth free action, $n>2$. Then μ is a smooth suspension of a smooth action on S^{2n-1}.*

This follows from Corollary 1 using the smoothing theory of PL manifolds.

If we consider the problem of comparing actions in the different categories, we first note as above that using the theorem of Kirby-Siebenmann, since p is odd there is a 1-1 correspondence between PL and topological actions on spheres of dimension 5 or more. The question of smoothing PL actions can be completely solved using results of Sullivan on the homotopy of G/PL and its relation to G/O in the "world of odd primes." In particular for $p=3$ or 5 every free PL Z_p action can be smoothed. However for $p=7$ there is a free PL action of Z_p on S^9 which cannot be smoothed, namely the suspension of a Z_p action on the generator Σ^7 of Γ^7, which may be constructed using the equation for the Brieskorn sphere $z_0^5+z_1^3+z_2^2+z_3^2+z_4^2=0$, $||z||=1$, and the action

$$T(z_0, \cdots, z_4) = (\lambda^6 z_0, \lambda^{10} z_1, \lambda^{15}(z_2, z_3, z_4)), \quad \text{where } \lambda^7 = 1.$$

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