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EXISTENCE AND UNIQUENESS FOR NONLINEAR NEUTRAL-DIFFERENTIAL EQUATIONS[1]

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ABSTRACT. Fixed point theorems are used to prove existence and uniqueness of the C^1 solution of the initial-value problem for a functional-differential equation of neutral type.

1. **Introduction.** In this paper we consider the initial-value problem (IVP) for the functional-differential equation of neutral type

$$x'(t) = f(t, x(t), x(g(t, x(t))), x'(h(t, x(t)))), \tag{1}$$

with the initial condition

$$x(0) = x_0. \tag{2a}$$

Here $f(t, x, y, z)$, $g(t, x)$ and $h(t, x)$ are continuous functions with $g(0, x_0) = h(0, x_0) = 0$. We assume further that the algebraic equation $z = f(0, x_0, x_0, z)$ has a real root z_0, and we require that

$$x'(0) = z_0. \tag{2b}$$

Existence theorems for IVP's for equation (1) have been proved by R. D. Driver [1] for the case where $h(t, x) < t$, and recently by V. P. Skripnik [2] under the hypotheses that f is sufficiently small, $h(t, x)$ is independent of x, and f is linear in the argument $x'(h(t))$. Our existence theorem requires none of these hypotheses. Under some additional conditions we obtain a local uniqueness theorem, and obtain as a corollary a result on existence of continuous solutions of certain nonlinear functional equations.

2. **Existence.** Let $\alpha > 0$ and let $J = [-\alpha, \alpha]$. We shall make the following assumptions concerning the IVP (1)–(2a)–(2b):

(i) $f(t, x, y, z)$ is continuous in some region in R^4 containing

$$P = \{(t, x, y, z): |t| \leqq \alpha, |x - x_0| \leqq \beta, |y - x_0| \leqq \beta, |z| \leqq M\}$$

where α, β and $M > |z_0|$ are positive constants. We assume that $\alpha \leqq \beta/M$ and that $\sup_{(t,x,y,z)\in P} |f(t, x, y, z)| \leqq M$.

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(ii) $g(t, x)$ and $h(t, x)$ are continuous in the projection $\tilde{R}$ of P into the (t, x) space; g and h both map $\tilde{R}$ into J, with $g(0, x_0) = h(0, x_0) = 0$, and $h(t, x)$ satisfies the Lipschitz condition

$$|h(t_1, x_1) - h(t_2, x_2)| \leqq k_1 |t_1 - t_2| + k_2 |x_1 - x_2|$$

for all (t_1, x_1), $(t_2, x_2) \in R$, where k_1 and k_2 are nonnegative constants with $k_1 + k_2 M \leqq 1$.

(iii) The function $f(t, x, y, z)$ satisfies the Lipschitz condition

$$|f(t, x, y, z_1) - f(t, x, y, z_2)| \leqq L_z |z_1 - z_2|$$

for all (t, x, y, z_1), $(t, x, y, z_2) \in P$, where $L_z < 1$.

The Schauder fixed-point theorem yields

THEOREM 1. *Under the hypotheses* (i)–(iii), *the IVP* (1)–(2a)–(2b) *has at least one solution which is continuously differentiable on J.*

3. **Uniqueness.** In case $h(t, x)$ is independent of x, we obtain the following theorem:

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *suppose that:*

(iv) *$h(t, x)$ is independent of x;*

(v) *f and g satisfy the Lipschitz conditions:*

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)|$$
$$\leqq L\{|x_1 - x_2| + |y_1 - y_2|\} + L_z |z_1 - z_2|$$

where L and L_z are nonnegative constants, with $L_z < 1$;

$$|g(t, x_1) - g(t, x_2)| \leqq L_g |x_1 - x_2|$$

with L_g a nonnegative constant, uniformly in their respective domains.

Then there exists γ_0, $0 < \gamma_0 \leqq \alpha$, such that there is a unique continuously differentiable solution of the IVP (1)–(2a)–(2b) *on the interval* $[-\gamma_0, \gamma_0]$.

The proof follows from the contraction mapping principle.

4. **Nonlinear functional equations.** As a corollary to our existence and uniqueness results, we note that if $f(t, x, y, z)$ is independent of x and y, and $h(t, x)$ is independent of x, the problem (1)–(2b) has the form of the functional equation

$$z(t) = f(t, z(h(t))), \tag{3}$$

$$z(0) = z_0, \tag{4}$$

where z_0 is a root of $z = f(0, z)$. Theorems 1 and 2 then yield at once:

THEOREM 3. *Let $f(t, z)$ be continuous in some region in R^2 containing $P_1 = \{t: |t| \leqq \alpha, \ |z| \leqq M\}$, where α and M are positive constants such that* $\sup_{(t,z)\in P_1} |f(t, z)| < M$, *and $M > |z_0|$ where z_0 is a real root of $z = f(0, z)$. Let f satisfy the Lipschitz condition $|f(t, z_1) - f(t, z_2)| \leqq L_z |z_1 - z_2|$ for all (t, z_1), $(t, z_2) \in P_1$, with $L_z < 1$. Let $h(t)$ be continuous for $|t| \leqq \alpha$, $h(0) = 0$, and $|h(t_1) - h(t_2)| \leqq |t_1 - t_2|$ for $t_1, t_2 \in [-\alpha, \alpha]$.*

Then the problem (3)–(4) *has at least one continuous solution on $[-\alpha, \alpha]$, and this is the unique continuous solution on this interval if α is sufficiently small.*

REFERENCES

1. R. D. Driver, *A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics*, Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, New York, Academic Press, 1963, pp. 474–484. MR **26** #4008.

2. V. P. Skripnik, *On some systems with deviating argument of neutral type*, Izv. Vysš. Učebn. Zaved. Matematika **1968**, no. 8 (75), 80–87. (Russian)

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