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A CORRESPONDENCE THEOREM FOR PROJECTIVE MODULES AND THE STRUCTURE OF SIMPLE NOETHERIAN RINGS[1]

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One of the tasks confronting ring theorists is the classification problem for simple noetherian rings. The Goldie theorems [1958], [1960] and the Lesieur-Croisot theorems [1959], provided the first structure theory for the non-artinian ones; and more generally, semiprime noetherian rings. The author [1964] showed that every simple right noetherian ring[3] is isomorphic to the endomorphism ring of a torsionfree module of finite rank U over a right Ore domain B, and Hart [1967] showed that U could be chosen to be finitely gen-

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[3] More generally, any simple ring with a uniform right ideal.

erated projective over B. These theorems proved to be simple consequences of a combination of a theorem of Goldie [**1958**], and a theorem of Morita [**1958**]. (Cf. Faith [**1967**].)

Now let A be any ring isomorphic to $\text{End}_B U$ where U is finitely generated projective and faithful over a ring B. The results just stated pose the following problem: characterize B in order that A is simple, respectively right noetherian. First, a correspondence theorem for projective modules is proved (Theorem 2) which enables one to give the characterization stated in Corollary 3. As a consequence, simple noetherian rings may be characterized as follows:

(A) THEOREM. *A ring A is simple and right noetherian if and only if $A \approx \text{End}_B U$, where U is finitely generated and projective over a right Ore domain B with just the ideals 0, $T = \text{trace}_B U$, and B, and satisfying the a.c.c. on idempotent right ideals contained in T (possibly $T = B$).*

(B) PROPOSITION. *When* (A) *holds, then B can be chosen such that the center C of B is a field isomorphic to the center of A, and, moreover, $B = C + T$.*

(C) PROPOSITION. *When* (A) *holds, then there is a right Ore domain B', a finitely generated projective left B' module U', and an isomorphism $A \approx \text{End}_{B'} U'$ only if B and B' have the same right quotient field D, and B and B' are equivalent right orders of D.*

The effect of (A), (B), and (C) is to reduce a certain amount of the structure of a simple noetherian ring A to that of a right Ore domain B determined up to equivalence of orders in the right quotient field of D. Moreover, *B has at most one nontrivial ideal.* This is the best possible theorem in the sense that every such B which arises can be a simple ring only if A is right hereditary. (See §7, Remark 2.)

The correspondence theorem depends on the following lemma of possibly independent interest.

1. LEMMA. *If U is projective and faithful over B_0, then U is projective over the biendomorphism (= bicommutator) ring B. Let T_0 and T denote the respective traces of U in B_0 and B. Then any right ideal I of B is a rational extension (in the sense of Findlay-Lambek* [**1958**]) *of IT_0; in particular, B is contained in the maximal right (Johnson-Utumi) quotient ring of B_0. Furthermore, T_0 is a left ideal of B such that $TT_0 = T_0$ and $T_0B = T_0T = T$. Thus, there is a lattice isomorphism (right ideals of $B)T \to ($right ideals of $B_0)T_0$ (sending $I \mapsto IT_0$) which induces an isomorphism $T($ideals of $B)T \to T_0($ideals of $B_0)T_0$.*

The method of proof of the next result are those of Morita in formulations exposited by Chase, Schanuel and Bass. (See Bass [1962], [1968].)

2. Correspondence Theorem for Projective Modules. *Assume that U is a finitely generated projective and faithful left module over B, and let $A = \mathrm{End}_B U$. Then there is a lattice isomorphism*

$$\begin{aligned} &(\textit{right ideals of } B)T \to A\textit{-submodules of } U \\ (1) \qquad &I = IT \mapsto IU \\ &I(W) \leftarrow\!\shortmid W. \end{aligned}$$

where $T = \mathrm{trace}_B U$, and $I(W)$ is the least right ideal I of B such that $IU = W$. Furthermore, there is an isomorphism of multiplicative semi-groups

$$\begin{aligned} (2) \qquad &(\textit{ideals of } A) \to T(\textit{ideals of } B)T \\ &J \mapsto I(UJ). \end{aligned}$$

3. Corollary. *Under the same hypotheses as Theorem 2, A is simple if and only if T is the least ideal K of B such that $TK \neq 0$. Similarly, A is (semi)prime if and only if $TK = 0$ for any annihilator (resp. nilpotent) ideal K of B properly contained in T. Thus, if B is semiprime, then A is simple if and only if T is the least ideal of B. In this case, a right ideal I of B contained in T is idempotent if and only if $I = IT$. (1) then implies that A is right noetherian (artinian) if and only if B satisfies the a.c.c. (d.c.c.) on idempotent right ideals contained in T. When B is a right Ore domain, this implies the a.c.c. (d.c.c.) on principal right ideals of* End U_A.

The reduction of the structure of a simple ring A to a ring containing at most one nontrivial ideal T is accomplished by the following lemma.

4. Lemma. *If U is finitely generated and projective over a ring B, and if $T = \mathrm{trace}_B U$, then, for any subring R of B containing T, the canonical left R-module U is finitely generated and projective. This holds in particular for $R = C + T$, where C is the center of B.*

Thus, when C is a field, as is the case when $A = \mathrm{End}_B U$ is simple, then either R is simple, or else, by Corollary 3, T is the only nontrivial ideal of R.

Other corollaries of the correspondence theorem are:

5. Corollary. *A ring A is a simple ring with maximal right annulet if and only if $A \approx \mathrm{End}_B U$, where U is a finitely generated projective and*

faithful left module over an integral domain B such that $B = \text{End } U_A$, and $T = \text{trace}_B U$ is the least nonzero ideal of B.

This corollary was inspired by Koh's observation that in any ring R, with maximal right annulet I, the endomorphism ring B of R/I is a domain. When A has a uniform right ideal, then B is actually a right Ore domain, as Goldie [**1958**], [**1960**] observed. Cf. also Procesi [**1963**].

6. COROLLARY. *A ring A is a simple ring with a uniform right ideal if and only if $A = \text{End}_B U$, where U is a finitely generated projective and faithful left module over a right Ore domain B, and $T = \text{trace}_B U$ is the least ideal $\neq 0$. Furthermore, $C = \text{center } B$ is a field isomorphic to* center A, *U is finitely generated projective over $R = C + T$, and $A = \text{End}_R U$ canonically. Moreover, either B is simple, or else T is the only nonzero ideal of R.*

7. REMARKS. 1. By Lemma 1, the biendomorphism ring of U over B is also a right Ore domain, a fact which enables one to dispense with the requirement $B = \text{End } U_A$ of Corollary 5.

2. By Faith [**1967**], B will be simple if and only if U is a generator in B-mod, and hence finitely generated projective over A. Then, there is an equivalence mod-$B \approx$ mod-A. Since U is canonically isomorphic to a right ideal of A, the endomorphism ring of every uniform right ideal of A is simple if and only if every submodule of U is finitely generated and projective. Since U is a generator, this is equivalent to the requirement that A be right noetherian and hereditary.

3. If $B = \text{End } U_A$, and if V is any A-submodule of U, then there is a right ideal I of B such that $IU = V$, and then $S = \text{End } I_B$ is canonically isomorphic to End V_A. If D is the right quotient field of B, then $S \approx \{a \in D \mid aI \subseteq I\}$ canonically. This shows that S and B are equivalent orders in D. Moreover, B and $R = C + T$ are equivalent orders. Finally, any uniform right ideal of A is isomorphic to an A-submodule of U.

ADDED MARCH 2, 1971. Regarding Lemma 1, T. Kato independently showed that the biendomorphism ring of a projective faithful, indeed, of any torsionless faithful, module U is a rational extension of the ring, in an unpublished paper entitled "*U-dominant dimension and U-localization*" received through the mail today.

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