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STRUCTURE OF WITT RINGS, QUOTIENTS OF ABELIAN GROUP RINGS, AND ORDERINGS OF FIELDS

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1. **Introduction.** In 1937 Witt [9] defined a commutative ring $W(F)$ whose elements are equivalence classes of anisotropic quadratic forms over a field F of characteristic not 2. There is also the Witt-Grothendieck ring $WG(F)$ which is generated by equivalence classes of quadratic forms and which maps surjectively onto $W(F)$. These constructions were extended to an arbitrary pro-finite group, $\mathfrak{G}$, in [1] and [6] yielding commutative rings $W(\mathfrak{G})$ and $WG(\mathfrak{G})$. In case $\mathfrak{G}$ is the galois group of a separable algebraic closure of F we have $W(\mathfrak{G})=W(F)$ and $WG(\mathfrak{G})=WG(F)$. All these rings have the form $\boldsymbol{Z}[G]/K$ where G is an abelian group of exponent two and K is an ideal which under any homomorphism of $\boldsymbol{Z}[G]$ to $\boldsymbol{Z}$ is mapped to 0 or $\boldsymbol{Z}2^n$. If C is a connected semilocal commutative ring, the same is true for the Witt ring $W(C)$ and the Witt-Grothendieck ring $WG(C)$ of symmetric bilinear forms over C as defined in [2], and also for the similarly defined rings $W(C, J)$ and $WG(C, J)$ of hermitian forms over C with respect to some involution J.

In [5], Pfister proved certain structure theorems for $W(F)$ using his theory of multiplicative forms. Simpler proofs have been given in [3], [7], [8]. We show that these results depend only on the fact that $W(F)\cong\boldsymbol{Z}[G]/K$, with K as above. Thus we obtain unified proofs for all the Witt and Witt-Grothendieck rings mentioned.

Detailed proofs will appear elsewhere.

2. **Homomorphic images of group rings.** Let G be an abelian torsion group. The characters χ of G correspond bijectively with the homomorphisms ψ_χ of $\boldsymbol{Z}[G]$ into some ring A of algebraic integers generated by roots of unity. (If G has exponent 2, then $A=\boldsymbol{Z}$.) The minimal prime ideals of $\boldsymbol{Z}[G]$ are the kernels of the homomorphisms $\psi_\chi:\boldsymbol{Z}[G]\to A$. The other prime ideals are the inverse images under the ψ_χ of the maximal ideals of A and are maximal.

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THEOREM 1. *If M is a maximal ideal of $\boldsymbol{Z}[G]$ the following are equivalent*:

(1) *M contains a unique minimal prime ideal.*

(2) *The rational prime p such that $M \cap \boldsymbol{Z} = \boldsymbol{Z}p$ does not occur as the order of any element of G.*

In the sequel K is a proper ideal of $\boldsymbol{Z}[G]$ and R denotes $\boldsymbol{Z}[G]/K$.

PROPOSITION 2. *The nil radical,* Nil R, *is contained in the torsion subgroup, R^t. We have $R^t = \text{Nil } R$ if and only if no maximal ideal of R is a minimal prime ideal and $R^t = R$ if and only if all maximal ideals of R are minimal prime ideals.*

THEOREM 3. *If p is a rational prime which does not occur as the order of any element of G, the following are equivalent*:

(1) *R has nonzero p-torsion.*

(2) *R has nonnilpotent p-torsion.*

(3) *R contains a minimal prime ideal $\overline{M}$ such that $R/\overline{M}$ is a field of characteristic p.*

(4) *There exists a character χ of G with $0 \neq \psi_\chi(K) \cap \boldsymbol{Z} \subset \boldsymbol{Z}p$.*

In addition, suppose now that G is an abelian q-group for some rational prime q. Then $\boldsymbol{Z}[G]$ contains a unique prime ideal M_0 which contains q.

COROLLARY 4. *The following are equivalent*:

(1) *R^t is q-primary.*

(2) *Let M be a maximal ideal of R which does not contain q, then M is not a minimal prime ideal.*

(3) *For all characters χ of G, $\psi_\chi(K) \cap \boldsymbol{Z} = 0$ or $\boldsymbol{Z}q^{n(\chi)}$.*

(4) *$K \subset M_0$ and all the zero divisors of R lie in $\overline{M}_0 = M_0/K$.*

THEOREM 5. *$R^t \subset \text{Nil } R$ if and only if $K \cap \boldsymbol{Z} = 0$ and one* (*hence all*) *of* (1), (2), (3), (4) *of Corollary* 4 *hold.*

THEOREM 6. *If K satisfies the conditions of Theorem* 5,

(1) *$R^t = \text{Nil } R$,*

(2) *$R^t \neq 0$ if and only if $\overline{M}_0$ consists entirely of zero divisors,*

(3) *R is connected.*

THEOREM 7. *The following are equivalent*:

(1) *For all characters χ we have $\psi_\chi(K) \cap \boldsymbol{Z} = \boldsymbol{Z}q^{n(\chi)}$.*

(2) *$R = R^t$ is a q-torsion group.*

(3) *$K \cap \boldsymbol{Z} = \boldsymbol{Z}q^n$.*

(4) *$M_0 \supset K$ and $\overline{M}_0$ is the unique prime ideal of R.*

These results apply to the rings mentioned in §1 with $q=2$. In particular, Theorems 5 and 6 yield the results of [5, §3] for Witt rings of formally real fields and Theorem 7 those of [5, §5] for Witt rings of nonreal fields.

By studying subrings of the rings described in Theorems 5–7 and using the results of [2] for symmetric bilinear forms over a Dedekind ring C and similar results for hermitian forms over C with respect to some involution J of C, we obtain analogous structure theorems for the rings $W(C)$, $WG(C)$, $W(C, J)$ and $WG(C, J)$. In particular, all these rings have only two-torsion, $R^t = \mathrm{Nil}\, R$ in which case no maximal ideal is a minimal prime ideal or $R^t = R$ in which case R contains a unique prime ideal. The forms of even dimension are the unique prime ideal containing two which contains all zero divisors of R. Finally, any maximal ideal of R which contains an odd rational prime contains a unique minimal prime ideal of R.

3. **Topological considerations and orderings on fields.** Throughout this section G will be a group of exponent 2 and $R = \boldsymbol{Z}[G]/K$ with K satisfying the equivalent conditions of Theorem 5. The images in R of elements g in G will be written $\bar{g}$. For a field F let $\dot{F} = F - \{0\}$. Then $W(F) = \boldsymbol{Z}[\dot{F}/\dot{F}^2]/K$ with K satisfying the conditions of Corollary 4. In this case K satisfies the conditions of Theorem 5 if and only if F is a formally real field.

THEOREM 8. *Let X be the set of minimal prime ideals of R. Then*

(a) *in the Zariski topology X is compact, Hausdorff, totally disconnected.*

(b) *X is homeomorphic to* $\mathrm{Spec}(\boldsymbol{Q}\otimes_{\boldsymbol{Z}} R)$ *and $\boldsymbol{Q}\otimes_{\boldsymbol{Z}} R \cong C(X, \boldsymbol{Q})$ the ring of $\boldsymbol{Q}$-valued continuous functions on X where $\boldsymbol{Q}$ has the discrete topology.*

(c) *For each P in X we have $R/P \cong \boldsymbol{Z}$ and $R_{red} = R/\mathrm{Nil}(R) \subset C(X, \boldsymbol{Z}) \subset C(X, \boldsymbol{Q})$ with $C(X, \boldsymbol{Z})/R_{red}$ being a 2-primary torsion group and $C(X, \boldsymbol{Z})$ being the integral closure of R_{red} in $\boldsymbol{Q}\otimes_{\boldsymbol{Z}} R$.*

(d) *By a theorem of Nöbeling [4], R_{red} is a free abelian group and hence we have a split exact sequence*

$$0 \to \mathrm{Nil}(R) \to R \to R_{red} \to 0$$

of abelian groups.

Harrison (unpublished) and Lorenz-Leicht [3] have shown that the set of orderings on a field F is in bijective correspondence with X

when $R=W(F)$. Thus the set of orderings on a field can be topologized to yield a compact totally disconnected Hausdorff space.

Let F be an ordered field with ordering $<$, $F_<$ a real closure of F with regard to $<$, and $\sigma_<$ the natural map $W(F)\to W(F_<)$. Since $W(F_<)\cong \boldsymbol{Z}$ (Sylvester's law of inertia), Ker $\sigma_< = P_<$ is a prime ideal of $W(F)$. Let the character $\chi_< \in \mathrm{Hom}(\dot{F}/\dot{F}^2, \pm 1)$ be defined by

$$\begin{aligned}\chi_<(aF^2) &= 1 \quad \text{if } a>0,\\ &= -1 \quad \text{if } a<0.\end{aligned}$$

PROPOSITION 9. *For u in R the following statements are equivalent*:

(a) *u is a unit in R.*

(b) *$u\equiv \pm 1 \bmod P$ for all P in X.*

(c) *$u=\pm\bar{g}+s$ with g in G and s nilpotent.*

COROLLARY 10 (PFISTER [5]). *Let F be a formally real field and $R=W(F)$. Then u is a unit in R if and only if $\sigma_<(u)=\pm 1$ for all orderings $<$ on F.*

Let E denote the family of all open-and-closed subsets of X.

DEFINITION. Harrison's subbasis H of E is the system of sets

$$W(a) = \{P\in X \mid a \equiv -1 \pmod{P}\}$$

where a runs over the elements $\pm\bar{g}$ of R.

If F is a formally real field and $R=W(F)$ then identifying X with the set of orderings on F one sees that the elements of H are exactly the sets

$$W(a) = \{<\text{on } F \mid a<0\}, \qquad a\in\dot{F}.$$

PROPOSITION 11. *Regarding R_{red} as a subring of $C(X, \boldsymbol{Z})$ we have*

$$R_{red} = \boldsymbol{Z}\cdot 1 + \sum_{U\in H} \boldsymbol{Z}\cdot 2f_U$$

where f_U is the characteristic function of $U\subset X$.

Following Bel'skiĭ [1] we call $R=\boldsymbol{Z}[G]/K$ a *small Witt ring* if there exists g in G with $1+g$ in K. Note that for F a field, $W(F)$ is of this type.

THEOREM 12. *For a small Witt ring R the following statements are equivalent*:

(a) *$E=H$.*

(b) (*Approximation.*) *Given any two disjoint closed subsets Y_1, Y_2 of X there exists g in G such that $\bar{g}\equiv -1 \pmod{P}$ for all P in Y_1 and $\bar{g}\equiv 1 \pmod{P}$ for all P in Y_2.*

(c) $R_{red} = \boldsymbol{Z} \cdot 1 + C(X, 2\boldsymbol{Z})$.

COROLLARY 13. *For a formally real field F the following statements are equivalent:*

(a) *If U is an open-and-closed subset of orderings on F then there exists a in $\dot{F}$ such that $<$ is in U if and only if $a<0$.*

(b) *Given two disjoint closed subsets Y_1, Y_2 of orderings on F there exists a in $\dot{F}$ such that $a<0$ for $<$ in Y_1 and $a>0$ for $<$ in Y_2.*

(c) $W(F)_{red} = \boldsymbol{Z} \cdot 1 + C(X, 2\boldsymbol{Z})$.

PROPOSITION 14. *Suppose F is a field with $\dot{F}/\dot{F}^2$ finite of order 2^n. Then there are at most 2^{n-1} orderings of F.*

If F is a field having orderings $<_1, \cdots, <_n$ we denote by σ the natural map $W(F) \rightarrow W(F_{<_1}) \times \cdots \times W(F_{<_n}) = \boldsymbol{Z} \times \cdots \times \boldsymbol{Z}$ via $r \rightarrow (\sigma_{<_1}(r), \cdots, \sigma_{<_n}(r))$.

THEOREM 15. *Let $<_1, \cdots, <_n$ be orderings on a field F. Then the following statements are equivalent:*

(a) *For each i there exists a in $\dot{F}$ such that $a <_i 0$ and $0 <_j a$ for $j \neq i$.*

(b) *$\chi_{<_1}, \cdots, \chi_{<_n}$ are linearly independent elements of* $\operatorname{Hom}(\dot{F}/\dot{F}^2, \pm 1)$.

(c) $\operatorname{Im} \sigma = \{(b_1, \cdots, b_n) \mid b_i \equiv b_j \pmod 2$ *for all* $i, j\}$.

If F is the field $\boldsymbol{R}((x))((y))$ of iterated formal power series in 2 variables over the real field, F has four orderings, $W(F) = W(F)_{red}$ is the group algebra of the Klein four group, and the conditions of Theorem 15 fail.

COROLLARY 16. *Suppose F is a field with $\dot{F}/\dot{F}^2$ finite of order 2^n. If condition* (a) *of Theorem* 15 *holds for the orderings on F then there are at most n orderings on F.*

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