A CLASS OF PERFECT DETERMINANTAL IDEALS

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In recent years several authors [1], [2], [4], [13], [17], [18] have studied the special homological properties of ideals generated by the subdeterminants of a matrix or "determinantal" ideals. The question of whether the ideal of $m+1$ by $m+1$ minors of an r by s matrix is perfect if the grade is as large as possible, $(r-m)(s-m)$, has remained open, although the special cases $m=0$, 1, and $r-1$ $(r\leqq s)$ are known. The general result is Corollary 4 of Theorem 1. For purposes of the induction argument used to prove the theorem it is necessary to consider a larger class of ideals somewhat complicated to describe.

THEOREM 1. *Let R be a commutative Noetherian ring with identity. Let $M=(c_{ij})$ be an r by s matrix with entries in R. Let $H=(s_0, \cdots, s_m)$ be a strictly increasing sequence of nonnegative integers such that $s_0=0$, $s_m=s$, and $m<r$. Let n be an integer, $0\leqq n\leqq s$. Let $I=I_{H,n}=I_{H,n}(M)$ be the ideal of R generated by the $t+1$ by $t+1$ minors of the first s_t columns of M, $1\leqq t\leqq m$, and $c_{11}, \cdots, c_{1n}$. Let h be the least integer such that $s_h\geqq n$. Suppose that the grade of (i.e. the length of the longest R-sequence contained in) I is as large as possible, namely*

$$g = g_{H,n} = rs - (r + s)m + h + \frac{1}{2}m(m + 1) + s_1 + \cdots + s_{m-1}.$$

Then $I_{H,n}$ is perfect in the sense of Rees, that is, the homological (or projective) dimension of R/I over R is also equal to g.

COROLLARY 1. *If $I_{H,n}$ has grade $g_{H,n}$ then it is grade unmixed, i.e. the associated primes of $I_{H,n}$ all have grade $g_{H,n}$.*

COROLLARY 2. *If R is Cohen-Macaulay (locally), and $I_{H,n}$ has grade $g_{H,n}$, then $I_{H,n}$ is rank unmixed, i.e. the associated primes all have rank ($\equiv$ altitude) $g_{H,n}$; moreover, R/I is Cohen-Macaulay.*

COROLLARY 3. *The rank of any minimal prime of $I_{H,n}$ is at most $g_{H,n}$ (with no conditions on the grade of I).*

COROLLARY 4. *When $H=(0, 1, 2, \cdots, m-1, s)$ and $n=0$, $I_{H,n}$ is*

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the ideal of $m+1$ *by* $m+1$ *minors of* M, *and the foregoing results hold in this case. Here*, $g=(r-m)(s-m)$.

In the terminology of [5], ideals of the form $I_{H,n}$ are *generically perfect*. They are even *strongly* generically perfect, i.e. they give rise to grade sensitive generalized Koszul complexes. See [7], [14]. We are not indulging in unnecessary generality in considering the whole class $I_{H,n}$ instead of just the ideals of Corollary 4. Our inductive proofs fail for any smaller class of ideals.

THEOREM 2. *Let* K *be a Noetherian domain. Let* $R=K[x]=K[x_{ij}]$, *where* $X=(x_{ij})$ *is an* r *by* s *matrix of indeterminates over* K. *Then* $I_{H,n}(X)$ *is a perfect radical ideal of grade* $g_{H,n}$. *If* $n=s_t$, $1\leqq t\leqq m$, *then* $I_{H,n}$ *is prime.*

COROLLARY. *If* $n=s_t$, $1\leqq t\leqq m$, *and* K *is integrally closed, then* $K[x]/I$ *is an integrally closed domain.*

By Proposition 4 of [5], Theorem 1 follows at once from Theorem 2. In fact, we only need the cases where K is either the integers or a finite prime field. (The corollary to Theorem 2 is proved by reducing to the case where K is a field, so that R/I is Cohen-Macaulay, and, in particular, principal ideals are unmixed, and then demonstrating that the singular locus has sufficiently small dimension.)

We point out that ideals of minors occur as the "Fitting invariants" of a module and in various geometric contexts, e.g. as the ideals of "determinantal loci" [16]. The ideals $I_{H,0}$ or, briefly, I_H, arise in the solution of the second main problem of invariant theory (see [19, Chapter II]) for certain representations of products of $GL(t, K)$, K a field of characteristic 0, t varying. That is, for each H, I_H is the ideal of relations on a set of generating invariants for such a representation. Specifically, let U_t be a $t+1$ by t matrix of indeterminates, $1\leqq t\leqq m-1$, let U_m be an r by m matrix of indeterminates, and let V_t be a t by s_t-s_{t-1} matrix of indeterminates, $1\leqq t\leqq m$. Define an action of $G=\prod_{t=1}^{m} GL(t, k)$ on $K[u, v]$ as follows: given $(A_1, \cdots, A_m)$ in G, let it act by taking the entries of U_t into the corresponding ones of $A_{t+1}U_tA_t^{-1}$, $1\leqq t\leqq m-1$, the entries of U_m into those of $U_mA_m^{-1}$, and the entries of V_t into those of A_tV_t, $1\leqq t\leqq m$. This determines an automorphism of $K[u, v]$. Consider the matrix X obtained by concatenating the product matrices $U_mU_{m-1}\cdots U_1V_1$, $U_mU_{m-1}\cdots U_2V_2, \cdots, U_mU_{m-1}V_{m-1}$, and U_mV_m. The entries of X generate the ring of invariants of this action, and the ideal of relations on the entries of X is precisely $I_H(X)$.

In the special case where I is the ideal of all $m+1$ by $m+1$ minors, matters can be made much simpler. Let U and V be r by m and m by s matrices of indeterminates, respectively, and define an action of $\mathrm{GL}(m, K)$ by $U \mapsto UA^{-1}$, $V \mapsto AV$. The entries of UV generate the ring of invariants, and the ideal of relations on the entries of $X = UV$ is precisely the ideal of $m+1$ by $m+1$ minors of X.

Our original proof of Theorem 2 (which required that K contain the rationals) utilized the fact that a product of copies of $\mathrm{GL}(t, K)$ is linearly reductive (as a linear algebraic group) for fields K of characteristic 0. We proved facts about the ideal structure in $K[X]/I_H$ by "lifting" the question to $K[u, v]$. The existence of the Reynolds operator implies that ideals expanded from the ring of invariants and then contracted back contract to themselves, making this procedure possible. See [**6**, pp. 116, 146, and 156] or [**11**, pp. 24–30]. This gave a proof of Theorem 1 when R contains the rationals. To get the general result, we sought and finally found arguments independent of invariant theory.

However, our original technique suggests that the theory of linear algebraic groups may have deep implications for the study of homological properties of ideals. It is natural to examine other representations of the classical groups on polynomial rings, and to ask whether the rings of invariants are Cohen-Macaulay (which is equivalent to asking whether the ideals of relations occurring are perfect). We have some special results of this type in addition to Theorem 2.

Our proof of perfection in Theorem 2, unlike that given for the maximal minor case in [**4**], does not depend on constructing an explicit resolution of R/I. Instead, it has some connection with the original proof of the maximal minor case given in [**2**]. We first show inductively that the ideals $I_{H,n}(X)$ are all radical. We next exhibit a generic point for $I_{H,n}$ if $n = s_t$, proving primality in this case. The proof of perfection then proceeds by Noetherian induction. If $n = s$, we can reduce r by one and change n to 0. If $n = s_t$, $t < m$, $x = x_{1,n+1}$ is not a zero divisor on $I_{H,n}$, and the perfection of $I_{H,n}$ follows from that of $I_{H,n} + (x) = I_{H,n+1}$. For $n \neq s_t$, any t, we show that $I_{H,n} = P \cap Q$, where P, Q and $P+Q$ are larger ideals of the form $I_{H',n'}$, and the grade of $P+Q$ is one more than the grades of $P \cap Q$, P and Q (which are equal). The result then follows from an easy homological lemma which asserts that under these circumstances, if P, Q and $P+Q$ are all perfect then so is $P \cap Q$. The details will appear in [**8**].

Finally, we note that the ideals $I_{H,n}$ are not, in general, complete intersections (cannot be generated by R-sequences): in fact, the generating sets specified are often minimal. If I is the ideal of maximal minors of $X = (x_{ij})$, the ring $K[x]/I$ is not even Gorenstein (see [**3**]).

Moreover, the "type" of ideal occurring in Corollary 4 differs as m varies. E.g. it is not difficult to show (by examining first Betti numbers) that, in general, the grade 4 ideal of 3 by 3 minors of a 4 by 4 matrix is not the grade 4 ideal of r by r minors of an r by $r+3$ matrix for any r.

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