FOLIATIONS AND NONCOMPACT TRANSFORMATION GROUPS

BY MORRIS W. HIRSCH

Communicated by Stephen Smale, November 20, 1969

Introduction. Let G be a Lie group and M a compact C^∞ manifold. In [2] Anosov actions of G on M are defined and proved to be structurally stable.

In this announcement we are concerned with the foliation $\mathfrak{F}$ of M defined by the orbits of G. Under the assumption that G is connected, $\mathfrak{F}$ is C^1 stable (3). If G is connected and nilpotent, G has a compact orbit (4). If G is merely solvable, however, there may be no compact orbit. In fact it can happen that no foliation C^0 close to $\mathfrak{F}$ has a compact leaf (8). Upper bounds for the number of compact orbits of given type are found (9). In (7) we discuss the intersection of certain nilpotent subgroups of a Lie group S with conjugates of a uniform discrete subgroup of S.

Hyperbolic automorphisms of foliations. A *k-foliation* $\mathfrak{F}$ of M is a function assigning to each $x \in M$ the image $\mathfrak{F}_x$ of a C^2 injective immersion $V_x \to M$ of a connected k-dimensional manifold V_x. We require that the *leaf* $\mathfrak{F}_x$ contain x, and that the function $T\mathfrak{F}: M \to G_k(M)$ assigning to $x \in M$ the tangent plane to V_x at x be C^1; here $G_k(M)$ is the manifold of k-planes tangent to M. Equivalently, $T\mathfrak{F}$ is a completely integrable C^1 field of k-planes, and $\mathfrak{F}_x$ is the maximal integral submanifold through x. Thus $\{\mathfrak{F}_x\}_{x\in M}$ is a partition of M. The set $\boldsymbol{F}_k(M)$ of all k-foliations of M inherits the C^0 and C^1 topologies from the set of C^1 maps $M \to G_k(M)$. We also use $T\mathfrak{F}$ to denote the bundle of k-planes tangent to the leaves.

If $\mathfrak{F}$, $\mathfrak{G} \in \boldsymbol{F}_k(M)$, a *homeomorphism* $h: \mathfrak{F} \to \mathfrak{G}$ is a homeomorphism of M taking each leaf of $\mathfrak{F}$ onto a leaf of $\mathfrak{G}$. We call $\mathfrak{F}$ C^1 *stable* if it has a C^1 neighborhood $N \subset \boldsymbol{F}_k(M)$ of foliations homeomorphic to $\mathfrak{F}$.

An *automorphism* g of $\mathfrak{F}$ is a C^1 diffeomorphism of M which is a homeomorphism $\mathfrak{F} \to \mathfrak{F}$. We call g *hyperbolic* if there exists a splitting $TM = E_+ \oplus E_- \oplus T\mathfrak{F}$ invariant under Tg, and such that the following condition holds. For some (and hence any) Riemannian metric on M there exist constants $0 < \lambda < 1 < \mu$ and $n \in \boldsymbol{Z}_+$ such that if $X \in TM$ and $X \neq 0$, then

AMS 1969 *subject classifications.* Primary 2240, 5736; Secondary 5482, 3465, 3451, 3453.

Key words and phrases. Foliation, transformation group, compact orbit, structural stability.

$$\begin{array}{lll} & |Tg^n X| < \lambda |X| & \text{if } X \in E_-; \\ \mu |X| < |Tg^n X| & & \text{if } X \in E_+; \\ \mu |X| > |Tg^n X| > \lambda |X| & & \text{if } X \in T\mathfrak{F}. \end{array}$$

A *deformation* of $\mathfrak{F}$ is an automorphism g which is homotopic to the identity by a homotopy g_t such that $g_t(\mathfrak{F}_x) \subset \mathfrak{F}_x$ for all $x \in M$ and $0 \leqq t \leqq 1$.

(1) THEOREM. *A foliation of M which admits a hyperbolic deformation is C^1 stable.*

The proof uses the stable manifold theory of [2].

(2) THEOREM. *Let g be a deformation of $\mathfrak{F} \in F_k(M)$ such that for some Riemannian metric on M, the Jacobian of*

$$Tg \mid T\mathfrak{F}_x : T\mathfrak{F}_x \to T\mathfrak{F}_{gx}$$

is $\neq 1$ for all x. Then $\mathfrak{F}$ has a C^0 neighborhood $N \subset F_k(M)$ such that no element of N has a compact leaf.

The idea of the proof is that if $\mathcal{G} \in N$, some approximation to g will be an automorphism of $\mathcal{G}$ which preserves every leaf of $\mathcal{G}$ but changes its volume.

Anosov actions. Let G be a Lie group acting differentiably of class C^2 on M. Suppose the action is *locally free*, i.e., every isotropy group is discrete. Then the components of orbits define a foliation $\mathfrak{F}$, and every $g \in G$ is an automorphism of $\mathfrak{F}$. If g is a hyperbolic automorphism, we call g an *Anosov element.* If an Anosov element exists we call the action an *Anosov action*; and we also say G acts *hyperbolically.*

Let G_1 be the identity component of G.

Let $\mathfrak{F}$ denote the orbit foliation of an Anosov action of G on M.

(3) THEOREM. *If G/G_1 is finite, or if G_1 contains an Anosov element, then $\mathfrak{F}$ is C^1 stable.*

PROOF. Follows from (1).

(4) THEOREM. *Assume G is nilpotent and G/G_1 finite. Then G has a compact orbit. In fact if $A \subset G$ is a 1-parameter subgroup containing an Anosov element g, the nonwandering set of the A flow on M lies in the closure K of the union of the compact G orbits. If g is measure preserving on M, then $M = K$, and some orbit is dense.*

The proof uses [2] to obtain a subgroup $H \subset G$ containing an Anosov element, such that $H(p)$ is compact for some $p \in M$. Since G

is nilpotent, the normalizer N of H is bigger than H, and [2] is used to prove that $N_1(p)$ is compact. By taking successive normalizers we arrive at G.

Examples.

(5) EXAMPLE. If G acts hyperbolically on M and H acts locally freely and transitively on N, then $G\times H$ acts hyperbolically on $M\times N$.

(6) EXAMPLE. Let G be an analytic subgroup of a Lie group S. Call G *hyperbolically embedded* if for some $X\in\mathfrak{L}(G)$ (the Lie algebra of G), $\mathrm{ad}_G X$ has its spectrum on the imaginary axis, while $\mathrm{ad}_S X$ induces an endomorphism of $\mathfrak{L}(S)/\mathfrak{L}(G)$ having no spectrum on the imaginary axis. If $\Gamma\subset S$ is a uniform discrete subgroup, then G acts hyperbolically on S/Γ. For example, $S=SL(n, R)$ and G is the diagonal subgroup. Note that Borel [1] proved that every semisimple Lie group has a uniform discrete subgroup.

From (4) we obtain

(7) THEOREM. *Let G be a connected nilpotent group which is hyperbolically embedded in S. Let $\Gamma\subset S$ be a uniform discrete subgroup. Then the set $\{s\in S \mid G/G\cap s\Gamma s^{-1}$ is compact$\}$ is dense in S; thus G has a uniform discrete subgroup.*

(8) EXAMPLE. Let A be a 2×2 real matrix such that e^A has integer entires, and its eigenvalues are λ and λ^{-1}, $0<\lambda<1$. Let $\boldsymbol{R}$ act on $\boldsymbol{R}^2$ by $t(x)=e^{tA}(x)$. Let S be the semidirect product $\boldsymbol{R}^2\cdot\boldsymbol{R}$. Note that $\boldsymbol{Z}\subset\boldsymbol{R}$ leaves $\boldsymbol{Z}^2\subset\boldsymbol{R}^2$ invariant, and set $\Gamma=\boldsymbol{Z}^2\cdot\boldsymbol{Z}\subset\boldsymbol{R}^2\cdot\boldsymbol{R}$. Put $M^3=S/\Gamma$; then M^3 is compact. Topologically, M^3 is obtained from $(\boldsymbol{R}^2/\boldsymbol{Z}^2)\times I$ by identifying $(x+\boldsymbol{Z}^2)\times 0$ with $(e^A(x)+\boldsymbol{Z}^2)\times 1$. Let $L\subset\boldsymbol{R}^2$ be the λ eigenspace of e^A and let $G=L\cdot\boldsymbol{R}\subset\boldsymbol{R}^2\cdot\boldsymbol{R}$. Then G is a solvable nonnilpotent 2-dimensional Lie group acting hyperbolically on M^3. The element $(0, 1)\in G$ is an Anosov element. It follows from Theorem 2 that *no sufficiently small C^0 perturbation of the orbit foliation of G on S/Γ has a compact leaf.* More generally, this is the case if G is hyperbolically embedded in S as in (6), and $\mathrm{tr}(\mathrm{ad}_G X)\neq 0$.

In contrast to this phenomenon, Roussarie and Weil state that any locally free action of $\boldsymbol{R}^2$ can be C^0 approximated by an action having compact orbits; see [4].

Counting compact orbits. Fix a Haar measure $\boldsymbol{m}$ for G. If $\Gamma\subset G$ is a discrete subgroup such that G/Γ is compact, it is well known that G is unimodular. Hence $\Delta(\Gamma)$ depends only on the conjugacy class $[\Gamma]$ of Γ, where $\Delta(\Gamma)$ is the measure of G/Γ induced by $\boldsymbol{m}$.

Let G act on M. The *type* of an orbit is the conjugacy class of the isotropy group of any of its elements. Let $N(\Gamma)$ be the number of compact orbits of type $[\Gamma]$.

(9) THEOREM. *Given an Anosov action of G on M, there is a constant B such that $N(\Gamma) \leqq B^{\Delta(\Gamma)}$ for every uniform discrete subgroup $\Gamma \subset G$.*

This theorem generalizes results of K. Meyer [3] and M. Shub [5]. The proof imitates Shub's.

BIBLIOGRAPHY

1. A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122. MR **26** #3823.

2. M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, Bull. Amer. Math. Soc. **76** (1970), 1015–1019.

3. K. R. Meyer, *Periodic points of diffeomorphisms*, Bull. Amer. Math. Soc. **73** (1967), 615–617. MR **36** #4573.

4. H. Rosenberg, R. Roussarie, and D. Weil, *A classification of closed orientable 3-manifolds of rank two*, University of Paris, Orsay. (mimeographed) (1968).

5. M. Shub, *Periodic orbits of hyperbolic diffeomorphisms and flows*, Bull. Amer. Math. Soc. **75** (1969), 57–58. MR **38** #2815.

UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720.