NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AND THE GENERALIZED TOPOLOGICAL DEGREE

BY FELIX E. BROWDER

Communicated March 9, 1970

Introduction. It is our purpose in the present note to present a general existence theorem for noncoercive elliptic boundary value problems for operators of the form:

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \cdots, D^m u), \tag{1}$$

on closed subspaces V of the Sobolev space $W^{m,p}(G)$, G an open subset of R^n, $n \geqq 1$. This existence theorem is based upon an extension of the theory of the generalized topological degree for A-proper mappings of Banach spaces introduced in Browder-Petryshyn [8], [9], and, in particular, on an extension of the Borsuk-Ulam theorem to pseudo-monotone mappings T from a reflexive separable Banach space V to its conjugate space V^*.

To make a precise statement of our general existence theorem possible, we introduce the following notation: For a given $m \geqq 1$, we let ξ be the m-jet of a function u from R^n to R^s for some given $s \geqq 1$, i.e. $\xi = \{\xi_\alpha : |\alpha| \leqq m\}$, and set

$$\zeta = \{\zeta_\alpha : |\alpha| = m\}, \qquad \eta = \{\eta_\beta : |\beta| \leqq m-1\},$$

where each ξ_α, ζ_α, and η_β is an element of R^s. The set of all ξ of the above form is an Euclidean space R^{r_m}, and correspondingly, $\zeta \in R^{r_m'}$, $\eta \in R^{r_{m-1}}$.

For each α, A_α is assumed to be a function from $G \times R^{r_m}$ to R^s satisfying the following conditions:

Assumptions on $A(u)$: (1) $A_\alpha(x, \xi)$ *is measurable in* x *for fixed* ξ *and continuous in* ξ *for fixed* x. *For a given* p *with* $1 < p < \infty$, *there exists a constant* c *such that*

$$|A_\alpha(x, \xi)| \leqq c\left((1 + \sum_{|\beta| \leq m} |\xi_\beta|^{p_{\alpha\beta}}\right)$$

with $p_{\alpha\beta} \leqq (p-1)$ *for* $|\alpha| = |\beta| = m$, *and*

AMS 1969 *subject classifications*. Primary 3547, 3536, 4780, 4785; Secondary 5536.

Key words and phrases. Nonlinear elliptic boundary value problems, generalized topological degree, Sobolev space, coercive, pseudomonotone, Borsuk-Ulam theorems, limit of A-proper mappings.

$$p_{\alpha\beta} < \frac{np + p(m - |\alpha|) - n}{n - p(m - |\beta|)}, \qquad \text{if } m - \frac{n}{p} \leqq |\alpha| \leqq m,$$

$$m - \frac{n}{p} \leqq |\beta| \leqq m,$$

$$|\beta| + |\alpha| < 2m,$$

$$p_{\alpha\beta} \leqq \frac{np}{n - p(m - |\beta|)}, \qquad \text{if } |\alpha| < m - \frac{n}{p},$$

$$m - \frac{n}{p} \leqq |\beta| \leqq m.$$

(2) *If* $\xi = (\zeta, \eta)$, *then for each* x *in* G, η *in* $R^{r_{m-1}}$, ζ *and* ζ' *in* R^{r_m} *with* $\zeta \neq \zeta'$,

$$\sum_{|\alpha|=m} \langle A_\alpha(x, \zeta, \eta) - A_\alpha(x, \zeta, \eta), \zeta_\alpha - \zeta'_\alpha\rangle > 0,$$

(*where* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* R^s).

(3) *For each* γ *and* γ' *in* $R^{r'_m}$,

$$\sum_{|\alpha|=m} \langle A_\alpha(x, \zeta, \eta) - \gamma_\alpha, \zeta_\alpha - \gamma'_\alpha\rangle \to \infty \qquad (|\zeta| \to \infty),$$

uniformly for bounded η.

Let $W^{m,p}(G)$ be the Sobolev space of s-vector functions u such that u and all its derivatives $D^\alpha u$ for $|\alpha| \leqq m$ lie in $L^p(G)$ where p is the exponent involved in the inequalities of Assumption (1). Then for any u and v in $W^{m,p}(G)$, we may define the generalized Dirichlet form corresponding to the representation (1) by:

$$a(u, v) = \sum_{|\alpha| \leqq m} (A_\alpha(\xi(u)), D^\alpha v), \tag{2}$$

where

$$\xi(u) = \{D^\alpha u : |\alpha| \leqq m\}, \qquad A_\alpha(\xi(u))(x) = A_\alpha(x, \xi(u)(x)),$$

$$(w, v) = \int_G \langle w(x), u(x)\rangle dx, \qquad \text{(integration with respect to Lebesgue } n\text{-measure).}$$

THEOREM 1. *Let* G *be an open subset of* R^n *with* G *bounded and the Sobolev Imbedding Theorem valid on* G (*i.e.* G *satisfies mild smoothness conditions on its boundary*). *Let* $A(u)$ *be a quasilinear elliptic operator of order* $2m$ *on* G *of the form*

$$A(u) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha(\xi(u)), \tag{1}$$

where the coefficient functions A_α *satisfy Assumptions* (1), (2), *and* (3) *above. Suppose that* $A(u)$ *is odd in* u, *i.e.* $A_\alpha(x, -\xi) = -A_\alpha(x, \xi)$ *for each* α *and all* x *in* G, ξ *in* R^{r_m}. *For each* w *in* V^*, *the dual space of a closed subspace* V *of* $W^{m,p}(G)$, *consider the problem of finding* u *in* V *such that* $a(u, v) = (w, v)$ *for all* v *in* V. *Suppose that there exists a continuous function* $\phi: R^+ \to R^+$ *such that for each solution* u *of this problem for any* w *in* V^*,

$$\|u\|_V = \|u\|_{W^{m,p}(G)} \leqq \phi(\|w\|_{V^*}). \tag{3}$$

Then for each w *in* V^*, *there exists at least one solution* u *in* V *of the problem*: $a(u, v) = (w, v)$ *for all* v *in* V.

We have used the notation (w, v) in Theorem 1 to denote the pairing between w in V^* and u in V.

THEOREM 2. *Let* G *be a bounded, smoothly bounded open set in* R^n *(as in Theorem* 1), *and consider a one-parameter family of operators* $A_t(u)$, $t \in [0, 1]$, *where for each* t,

$$A_t(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(\xi(u); t) \tag{4}$$

and the coefficient functions are continuous in t, *uniformly for bounded* ξ *and all* x *outside a null set in* G. *For each* t, *we take the generalized Dirichlet form*

$$a_t(u, v) = \sum_{|\alpha| \leq m} (A_\alpha(\xi(u); t), D^\alpha v), \tag{5}$$

where we assume that $A_t(u)$ *satisfies Assumptions* (1), (2), (3) *for each* t *in* $[0, 1]$. *Suppose that* $A_1(u)$ *is odd, and that there exists a continuous function* $\phi: R^+ \to R^+$ *such that if* $a_t(u, v) = (w, v)$ *for some* w *in* V^*, u *in* V, t *in* $[0, 1]$ *and all* v *in* V, *then* $\|u\|_V \leqq \phi(\|w\|_{V^*})$.

Then the problem: $a_0(u, v) = (w, v)$ *for all* v *in* V; *has a solution* u *in* V *for each* w *in* V^*.

Theorem 2 includes Theorem 1 as the special case in which $A_t(u) = A(u)$ for all t in $[0, 1]$. It also includes the standard existence theorem for $A(u)$ in which the Dirichlet form $a(u, v)$ is assumed to be coercive, i.e.

(6) *There exists* $c: R^+ \to R^1$ *with* $c(r) \to \infty$ *as* $r \to \infty$ *such that* $a(u, u) \geqq c(\|u\|_V)\|u\|_V$.

Indeed, if $A(u)$ is coercive, and if we set $A_t(u) = A(u) - tA(-u)$ for t in $[0, 1]$, then $A_0(u) = A(u)$, $A_1(u)$ is odd, the Assumptions (1), (2), and (3) hold for every $A_t(u)$, while since $a_t(u, u) = a(u, u) - ta(-u, u) = a(u, u) + ta(-u, -u)$, it follows that

$$a_t(u, u) \geqq (1 + t)c(\|u\|_V)\|u\|_V \geqq c(\|u\|_V)\|u\|_V$$

provided that $\|u\|_V > R$, where $c(r) > 0$ for $r > R$. Suppose that for some u in V, w in V^* and t in $[0, 1]$, we have

$$a_t(u, v) = (w, v) \qquad (v \in V).$$

Then:

$$c(\|u\|_V)\|u\|_V \leqq a_t(u, u) = (w, u) \leqq \|w\|_{V^*}\|u\|_V,$$

and as a consequence $c(\|u\|_V) \leqq \|w\|_{V^*}$ if $u = 0$. If we set $\phi(s) = \sup\{r : c(r) \leqq s\}$, it follows that $\|u\|_V \leqq \phi(\|w\|_{V^*})$ and by Theorem 2, the equation $a(u, v) = (w, v)$ $(v \in V)$, has a solution u in V for each w in V^*.

Existence theorems for elliptic boundary problems of this type were first obtained by Višik [15] using compactness arguments and a priori estimates on $(m+1)$st derivatives. Monotonicity arguments were first applied to these problems in Browder [2], [3], using the basic existence theorem for monotone maps from a reflexive Banach space V to V^* proved independently by Browder [2] and Minty [12]. The existence theorem in the coercive case was extended to elliptic operators $A(u)$ satisfying Assumptions (1), (2), and (3) by Leray-Lions [11]. Borsuk-Ulam theorems for monotone and semimonotone operators in infinite dimensional Banach spaces were first derived by Browder [4], [5], and were first applied to odd, homogeneous, elliptic operators satisfying strong monotonicity conditions by Pohožaev [14]. Theorem 1 was first obtained under a stronger hypothesis (3)′ rather than (3) in Browder [6], together with Assumptions (1) and (2) on $A(u)$. This is as follows:

(3)′ *There exist continuous functions* $k(\eta)$, $k_0(\eta) > 0$ *such that*

$$\sum_{|\alpha| \leq m} \langle A_\alpha(x, \zeta, \eta)\zeta_\alpha\rangle \geqq k_0(\eta)\,|\zeta|^p - k(\eta),$$

for all x *in* G, ζ *in* $R^{r'_m}$, η *in* $R^{r_{m-1}}$.

1. Proofs of Theorems 1 and 2 rest upon general results concerning two classes of nonlinear mappings of monotone type from a reflexive Banach space V to its conjugate space V^*.

DEFINITION 1. *Let V be a Banach space, V^* its conjugate space, T a mapping from V to V^*, Then:*

(a) *T is said to be pseudomonotone if for any sequence $\{u_j\}$ in V with u_j converging weakly to u in V such that* $\limsup(Tu_j, u_j - u) \leqq 0$, *it follows that for any v in V,* $\liminf\,(Tu_j, u_j - v) \geqq (Tu, u - v)$.

(b) *T is said to satisfy condition $(S)_+$ if for any sequence u_j in V with*

$\{u_j\}$ converging weakly to u in V for which $\lim(Tu_j, u_j-u) \leqq 0$, *it follows that u_j converges strongly to u in V.*

PROPOSITION 1. *Suppose that A satisfies Assumption* (1). *Then there exists a continuous bounded mapping T of V into V^* for a given closed subspace V of $W^{m,p}(G)$ such that for all u and v of V,* $(Tu, v) = a(u, v)$. *If $A(u)$ satisfies Assumptions* (2) *and* (3), *T is pseudomonotone. If $A(u)$ satisfies Assumptions* (2) *and* (3)′, *then T satisfies condition* $(S)_+$.

The proof of Proposition 1 is given in §1 of [7], and Appendix to §1. Pseudomonotonicity was first defined by Brézis in [1] (though our definition differs slightly from his in considering only sequences rather than filters). The condition $(S)_+$ was first defined in connection with the study of nonlinear eigenvalue problems in Browder [6] and is studied in detail in Browder [7], [8].

THEOREM 3. *Let V be a reflexive separable Banach space, T a mapping of V into V^* which is pseudomonotone, bounded on bounded sets, and continuous from each finite dimensional subspace of V to the weak topology of V^*. Then*:

(a) *If T is an odd mapping outside of some ball around the origin and if $T^{-1}(B)$ is bounded for any bounded subset B of V^*, then $R(T)$, the range of T, is all of V^*.*

(b) *If $\{T_t\}$ is a family of bounded, pseudomonotone, finitely continuous mappings from V to V^* which is continuous in t uniformly on bounded subsets of V, with $T_0 = T$, T_1 odd outside some ball, and if there exists a function $\phi: R^+ \to R^+$ such that $T_t(u) = w$ implies that*

$$\|u\| \leqq \phi(\|w\|) \qquad (t \in [0, 1]),$$

then $R(T) = V^*$.

Theorem 3 and Proposition 1 together imply the validity of Theorems 1 and 2. Theorem 3 follows from an extension to the class of pseudomonotone mappings from V to V^* of the theory of the generalized degree defined for A-proper mappings of Banach spaces in Browder-Petryshyn [9], [10] and applied to mappings T from a reflexive V to V^* satisfying condition (S) in Chapter 17 of Browder [8]. The basic facts are summarized in the following theorem:

THEOREM 4. *Let V be a reflexive separable Banach space, V^* its conjugate space. Let T be a mapping from V to V^* which is finitely continuous from V to V^* (i.e. continuous from each finite dimensional subspace of V to the weak topology of V^*) and bounded (i.e. maps bounded subsets of V into bounded subsets of V^*). Then*:

(a) *If T is pseudomonotone, there exists a sequence $\{T_j\}$ of finitely continuous, bounded mappings, each satisfying condition $(S)_+$, which converges to T uniformly on every bounded subset of V.*

(b) *If T satisfies condition $(S)_+$, then T is A-proper in the following sense* [9], [10]: *If B is a closed ball of V, $\{V_j\}$ an increasing sequence of finite dimensional subspaces of V whose union is dense in V, and if for each j, u_j is an element of $V_j \cap B$ such that for a given element w of V^*,*

$$\|\phi_j^* T u_j - \phi_j^* w\|_{V_j^*} \to 0 \qquad (j \to \infty),$$

where ϕ_j is the injection map of V_j into V, ϕ_j^* the projection map of V^* onto V_j^*, then there exists an infinite subsequence $\{u_{j(k)}\}$ converging strongly to an element u of B such that $T(u) = w$.

The proof of Theorem 4 is given in Chapter 17 of Browder [8]. The second property tells us that the generalized degree theory of Browder-Petryshyn [10] applies to mappings T satisfying the condition $(S)_+$ (for the details of this application, see [8]). The corresponding generalized degree theory for pseudomonotone maps follows from the convexity of the class of T satisfying $(S)_+$ and the following theorem whose proof will be published elsewhere:

THEOREM 5. *Let X and Y be Banach spaces, G a bounded open subset of X, and consider an oriented approximation scheme $\{(X_n, Y_n, P_n, Q_n)\}$ for mappings T of* $\mathrm{cl}(G)$ *into Y in the sense of* [10]. *Let Z be a convex family of A-proper mappings from* $\mathrm{cl}(G)$ *to Y with respect to the given approximation scheme. Let T be a mapping from* $\mathrm{cl}(G)$ *to Y which is the uniform limit on* $\mathrm{cl}(G)$ *of mappings T_j from the class Z. Then*:

(a) *For any sequence $\{T_j\}$ from Z converging to T, if w does not lie in* $\mathrm{cl}(T(\mathrm{bdry}(G)))$, *then* $\mathrm{Deg}(T_j, G, w)$ *is the same for all j sufficiently large and does not depend upon the choice of $\{T_j\}$. We denote this limit as* $\mathrm{Deg}(T, G, w)$.

(b) $\mathrm{Deg}(T, G, w)$ *is invariant under homotopy and weakly additive in the sense of Theorem* 1 *of* [10]. *If* $\mathrm{Deg}(T, G, w) \neq \{0\}$ *and if* $T(\mathrm{cl}(G))$ *is closed in Y, then w lies in* $T(\mathrm{cl}(G))$.

(c) *If T is odd in the sense of Theorem* 1 *of* [10], *then* $\mathrm{Deg}(T, G, 0)$ *consists only of odd integers, and* $\mathrm{Deg}(T, G, 0) \neq \{0\}$.

ADDED IN PROOF. Results closely related to Theorem 5 have also been obtained by P. M. Fitzpatrick in connection with his Rutgers Ph.D. dissertation.

BIBLIOGRAPHY

1. H. Brézis, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier Grenoble **18** (1968), 115–175.

2. F. E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc. **69** (1963), 862–874. MR **27** #6048.

3. ———, *Nonlinear elliptic boundary value problems*. II, Trans. Amer. Math. Soc. **117** (1965) 530–550. MR **30** #4054.

4. ———, *Fixed-point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1272–1276. MR **31** #2582.

5. ———, *Mapping theorems for noncompact nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 337–342. MR **31** #5113.

6. ———, *Nonlinear eigenvalue problems and Galerkin approximations*, Bull. Amer. Math. Soc. **74** (1968), 651–656. MR **37** #2043.

7. ———, *Existence theorems for nonlinear partial differential equations*, Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, R. I. (to appear).

8. ———, *Nonlinear operators and nonlinear equations of evolution*, Proc. Sympos. Pure Math., vol. 18, Amer. Math. Soc., Providence, R. I. (to appear).

9. F. E. Browder and W. V. Petryshyn, *The topological degree and Galerkin approximations for noncompact operators in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 641–646. MR **37** #4678.

10. ———, *Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces*, J. Functional Analysis **3** (1969), 217–245.

11. J. Leray and J. L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France **93** (1965), 97–107. MR **33** #2939.

12. G. J. Minty, *On a "monotonicity" method for the solution of nonlinear equations in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1038–1041. MR **28** #5358.

13. J. Neças, *Sur l'alternative de Fredholm pour les opérateurs non-linéaires avec applications aux problèmes aux limites*, Ann. Scuola Norm. Sup. Pisa (3) **23** (1969), 332–345.

14. S. I. Pohožaev, *The solvability of nonlinear equations with odd operators*, Funkcional. Anal. i Priložen. **1** (1967), no. 3, 66–73. (Russian) MR **36** #4396.

15. M. I. Višik, *Quasi-linear strongly elliptic systems of differential equations in divergence form*, Trudy Moskov. Mat. Obšč. **12** (1963), 125–184 = Trans. Moscow Math. Soc. **1963**, 140–208. MR **27** #6017.

UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637