HÖLDER AND L^p ESTIMATES FOR SOLUTIONS OF $\bar\partial u=f$ IN STRONGLY PSEUDOCONVEX DOMAINS[1]

BY NORBERTO KERZMAN

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1. **Introduction.** A recent result due to H. Grauert and I. Lieb [1] asserts that if G is a strongly pseudoconvex domain with smooth boundary, $G\subseteq C^n$, and if $f=\sum_{j=1}^n f_j\, d\bar z_j$ is a C^∞, (0, 1) form in G, $\bar\partial f=0$, f bounded, then the equation $\bar\partial u=f$ has a solution $u:G\to C$ such that $\sup_{z\in G}|u(z)|\leqq C\sup_{z\in G}|f(z)|$, where $|f(z)|=\sum_{j=1}^n|f_j(z)|$. Grauert and Lieb's theorem is proved by writing a solution u in the form $u(w)=\int_G\Omega(z,w)\wedge f(z)$, $w\in G$ and then estimating the kernel to obtain $\int_G|\Omega(z,w)|dz\leqq A<\infty$, A independent of $w\in G$. The kernel $\Omega(z,w)$ is the one constructed by E. Ramirez in [6] who employed it to obtain an integral representation formula for holomorphic functions. Ramirez' construction of $\Omega(z,w)$ involves the application of Cartan's theorem B for vector valued functions as well as a division theorem which he proves in [6].

We have found an alternate approach using Hörmander's L^2 estimates which yields a somewhat simpler proof: We first determine (Theorem L) a local solution by the same method as in Grauert and Lieb's paper. In this local case, however, a kernel $\Omega(z,w)$ can be written explicitly. Our passage from local to global then uses only Hörmander's L^2 estimates for the $\bar\partial$ problem. By this method we obtain a stronger result, namely a solution u satisfying a Hölder condition with any exponent α, $\alpha<1/2$, up to the boundary of G (Theorem 1). The method also yields (Theorem 2) solutions in L^p whenever $f\in L^p$, $1\leqq p\leqq\infty$; this is not an interpolation result even for $2\leqq p\leqq\infty$ (see remarks following Theorem 2).

As an application of Grauert-Lieb's theorem we prove (Theorem 3) that holomorphic functions which are continuous up to the boundary of G can be uniformly approximated on G by holomorphic functions defined in a neighborhood of $\bar G$. This result has been proved independently and at about the same time by I. Lieb [5] using the Ramirez integral formula.

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[1] The results of this note are part of the author's thesis at New York University, Courant Institute of Mathematical Sciences. The proofs will appear in full elsewhere.

2. In the sequel G stands for a strongly pseudoconvex domain $G \subseteq C^n$ with smooth C^4 boundary, i.e. G is open, G is compact and there exists an open neighborhood U of ∂G and a C^4 function $\lambda: U \to R$ such that $G \cap U = \{z \in U, \lambda(z) < 0\}$; λ is a strictly plurisubharmonic function, i.e. $\sum_{i,j=1}^n (\partial^2\lambda/\partial z_i \partial \bar{z}_j)(z)\ \mu_i \bar{\mu}_j \geqq A(z)|\mu|^2$ for all $z \in U$ and $\mu \in C^n$; here $A(z) > 0$. The gradient $\nabla\lambda(z) \neq 0$ in U.

THEOREM 1. *Let G be a strongly pseudoconvex domain $G \subseteq C^n$ with smooth C^4 boundary and let f be a C^∞, (0, 1) form defined in G, $\bar{\partial} f = 0$, f bounded. There exists a solution u of the equation $\bar{\partial} u = f$ in G such that*

$$(1) \qquad \sup_{w,w' \in G} \frac{|u(w) - u(w')|}{|w - w'|^\alpha} \leqq C_\alpha \sup_{w \in G} |f(w)|, \qquad \alpha < \tfrac{1}{2},$$

where α is any number $\alpha < 1/2$ and C_α is independent of f.

COROLLARY 1. *If G and f are as in Theorem 1, then there is a solution u of $\bar{\partial} u = f$ which is continuous up to the boundary of G (even though f is defined in G and need not be continuous in $\bar{G}$).*

REMARKS. The solution u we obtain depends linearly on f. The constant C_α is independent of G for small C^4 perturbations of G. It is well known that if $n = 1$, then (1) holds with any exponent $\alpha < 1$.

THEOREM 2. *Let G be as in Theorem 1, and let f be a C^∞, (0, 1) form in G, $\bar{\partial} f = 0$, $f \in L^p(G)$, $1 \leqq p \leqq \infty$. There exists a solution u of $\bar{\partial} u = f$ in G such that*

$$(2) \qquad \|u\|_{L^p(G)} \leqq c\|f\|_{L^p(G)}, \qquad 1 \leqq p \leqq \infty.$$

The remarks above also apply here, with C_α replaced by c; c is independent of p.

For $p = 2$, (2) are the well-known J. J. Kohn or L. Hörmander L^2 estimates [3], [4], [2]. The case $p = \infty$ is Grauert-Lieb's theorem. The intermediate cases $2 \leqq p \leqq \infty$ cannot be obtained by interpolation of these two known results because the operator which gives the solution u in Grauert-Lieb's paper differs from Kohn's and from Hörmander's. In Theorems 1 and 2 the operators T giving the solution $u = Tf$ all agree, so there is really only one operator T involved. However the computation which shows that T is continuous from L^2 to L^2 and from L^∞ to L^∞ also shows at the same time that T is continuous from L^p to L^p, $2 \leqq p \leqq \infty$.

3. **Proofs.** Both Theorems 1 and 2 are obtained from their "local" versions 1_L and 2_L (which are lumped together as Theorem L below) via an application of Hörmander's L^2 estimates for the $\bar{\partial}$ problem.

Let $B_a(q)$ denote the ball of center $q\in \mathbf{C}^n$ and radius a.

THEOREM L. *Let G and f be as in Theorem 2. There exist positive numbers a, c', C'_α, independent of f, such that in any set $G\cap B_a(q)$, $q\in\partial G$, the equation $\bar\partial u=f$ has a solution $u: G\cap B_a(q)\to C$ satisfying*

$$(2_L) \qquad \|u\|_{L^p(G\cap B_a(q))} \leqq c'\|f\|_{L^p(G)}, \qquad 1\leqq p\leqq\infty.$$

If $p=\infty$, u satisfies, in addition,

$$(1_L) \qquad \sup_{w,w'\in G\cap B_{a(q)}} \frac{|u(w)-u(w')|}{|w-w'|^\alpha} \leqq C'_\alpha\|f\|_{L^\infty(G)}, \qquad \alpha<1/2,$$

where α is any number, $\alpha<1/2$.

REMARK. The solution u is linear in f; a, c', C'_α are independent of G for small C^4 perturbations of G; a and c' are independent of p, $1\leqq p\leqq\infty$.

Theorem L is proved (see Introduction) by explicitly constructing a kernel $\Omega(z,w)$, $z\in B_{2a}(q)\cap G$, $w\in B_a(q)\cap G$, a small, which gives a solution u of the form

$$(3) \qquad u(w)=\int_{z\in G\cap B_{2a}(q)} \Omega(z,w)\wedge f(z), \qquad w\in G\cap B_a(q).$$

Then (1_L) and (2_L) are proved by direct (though nontrivial) estimations in (3).

Theorem L enables us to make quantitative a well-known extension trick (Lemma 1) which in turn implies Theorems 1 and 2 (using L^2 estimates for solutions of $\bar\partial u=f$).

LEMMA 1. *Let G be as in Theorem 2. There exists a (slightly bigger) strongly pseudoconvex domain $\hat G$, $G\subseteq\bar G\subseteq\hat G$, having the following property: for any form f as in Theorem 2 there exists a C^∞, (0, 1) form $\hat f$ in $\hat G$ and a C^∞ function $\chi: G\to C$ such that $\bar\partial\hat f=0$ in $\hat G$, $\hat f=f+\bar\partial\chi$ in G and*

$$(4) \qquad \|\hat f\|_{L^p(\hat G)} \leqq c''\|f\|_{L^p(G)}, \qquad 1\leqq p\leqq\infty,$$

$$(5) \qquad \|\chi\|_{L^p(G)} \leqq c''\|f\|_{L^p(G)}, \qquad 1\leqq p\leqq\infty.$$

If $p=\infty$, χ satisfies, in addition,

$$(6) \qquad \sup_{w,w'\in G} \frac{|\chi(w)-\chi(w')|}{|w-w'|^\alpha} \leqq C''_\alpha\|f\|_{L^\infty(G)}, \qquad \alpha<\tfrac{1}{2}.$$

The constants are independent of f; α is any number, $\alpha<1/2$.

PROOF OF THEOREMS 1 AND 2. We only consider here the case $2\leqq p\leqq\infty$. Let $\hat G$, $\hat f$ and χ be as above. Since $\hat G$ is pseudoconvex there is a $\hat{\mu}:\hat{G}\to \mathbf{C}$ such that $\bar{\partial}\hat{\mu}=\hat{f}$ in $\hat{G}$ and

$$(*)\qquad \|\hat{u}\|_{L^2(\hat{G})} \leqq K\|\hat{f}\|_{L^2(\hat{G})}.$$

See [2, p. 107]. Thus, $\bar{\partial}(\hat{u}-\chi)=f$ in G; $u=\hat{u}-\chi$ satisfies Theorems 1 and 2. For χ clearly does, and $\hat{u}$ satisfies

$$(7)\qquad \|\hat{u}\|_{L^p(G)} \leqq K_1[\|\hat{u}\|_{L^2(\hat{G})} + \|\bar{\partial}\hat{u}\|_{L^p(\hat{G})}], \qquad 1 \leqq p \leqq \infty,$$

$$(8)\qquad \sup_{w,w'\in G} \frac{|u(w)-u(w')|}{|w-w'|^{\alpha}} \leqq K_2[\|\hat{u}\|_{L^2(\hat{G})} + \|\bar{\partial}\hat{u}\|_{L^\infty(\hat{G})}], \qquad \alpha < 1.$$

The estimates (7) and (8) are valid for any smooth function $\hat{u}$ in $\hat{G}$ since $\bar{\partial}$ is elliptic; in this special case (i.e. for the operator $\bar{\partial}$) they can be easily checked directly. Since $\|\hat{u}\|_{L^2(\hat{G})}\leqq K'\|\hat{f}\|_{L^p(\hat{G})}$ (using (*) and $p\geqq 2$), application of (4) yields Theorems 1 and 2.

4. Uniform approximation of holomorphic functions.

See the introduction to this note, [7] and I. Lieb [5].

THEOREM 3. *Let $G\subseteq \mathbf{C}^n$ be a strongly pseudoconvex domain with smooth C^4 boundary. There exists an open set $\hat{G}\subseteq \mathbf{C}^n$, $G\subseteq\bar{G}\subseteq\hat{G}$ such that any continuous function $u:\bar{G}\to\mathbf{C}$ which is holomorphic in G can be uniformly approximated on $\bar{G}$ by holomorphic functions $\hat{u}$ defined in $\hat{G}$.*

PROOF. Cover ∂G by small balls $B_i=B_a(p_i)$, $i=1,\cdots,k$; shift $U_i=G\cap B_i$ in the direction of the outward normal n_i at p_i to obtain $U_i^\delta=U_i+\delta n_i$, $0<\delta$ small. The holomorphic functions $u_i^\delta: U_i^\delta\to\mathbf{C}$, $u_i^\delta(z)=u_i(z-\delta n_i)$ may not agree in $U_i^\delta\cap U_j^\delta$, $i\neq j$. Set $U_0^\delta=G$, $u_0^\delta=u$, and let G^δ be such that $G\subseteq\bar{G}\subseteq G^\delta\subseteq\bar{G}^\delta\subseteq\bigcup_{i=0}^k U_i^\delta$ and G^δ is strongly pseudoconvex with smooth boundary. Restrict u_i^δ to v_i^δ: $V_i^\delta\to\mathbf{C}$, where $V_i^\delta=U_i^\delta\cap G^\delta$. In G^δ consider the covering V_i^δ, $i=0,\cdots,k$, and the holomorphic cocycle $v_i^\delta=v_i^\delta-v_j^\delta$, v_{ij}^δ: $V_i^\delta\cap V_j^\delta\to\mathbf{C}$.

Theorem 3 is proved by solving a first Cousin problem with bounds: There exist holomorphic functions h_i^δ: $V_i^\delta\to\mathbf{C}$ such that $h_i^\delta-h_j^\delta=v_{ij}^\delta$ and

$$(9)\qquad \sup_{z\in V_i;i=0,\ldots,k} |h_i^\delta(z)| \leqq C \sup_{z\in V_i\cap V_j;i,j=0,\ldots,k} |v_{ij}^\delta(z)|,$$

where C is independent of δ. The holomorphic function $v^\delta=v_i^\delta-h_i^\delta=v_j^\delta-h_j^\delta$ is then globally defined in G^δ. When $\delta\to 0$ the uniform continuity of u in $\bar{G}$ and (9) yield $v^\delta\to u$ uniformly on $\bar{G}$; the functions v^δ are holomorphic in shrinking neighborhoods of $\bar{G}$. Finally each v^δ can be uniformly approximated on $\bar{G}$ by holomorphic functions defined in a fixed set $\hat{G}\supseteq\bar{G}$. (This is a well-known result.)

The functions h_i are obtained by application of Grauert-Lieb's theorem (i.e. Theorem 2 in case $p=\infty$) to the form $f=\bar{\partial}g_i^\delta=\bar{\partial}g_j^\delta$,

$g_i^\delta = \sum_{j=0}^k v_{ij}^\delta \phi_j^\delta$, where ϕ_j^δ correspond to a convenient partition of unity in $\bar{G}^\delta$; G^δ is chosen close to G in the C^4 sense.

(a) The Hölder condition (1) does not hold for exponents $\alpha > \frac{1}{2}$. It may also fail in polydiscs, even for exponents $< \frac{1}{2}$.

All three theorems above, as well as their proofs are valid also in case G is contained in a Stein manifold. Theorems 1 and 2 hold also for nonsmooth forms f; $\bar{\partial}$ is considered in distribution sense.

(b) G. Henkin had constructed a global kernel $\Omega(z, w)$ similar to Ramirez', and proposed a proof of approximation Theorem 3. See G. Henkin, *Integral representations of holomorphic functions in strongly pseudoconvex domains and certain applications*, Mat. Sb **78** (1969), 611–632, (Russian), specially footnote in p. 631.

I. Lieb has extended the result in [1] to the case of $(0, q)$ bounded smooth forms f in G, $\bar{\partial} f = 0$ obtaining a bounded solution u of $\bar{\partial} u = f$. See I. Lieb, *Beschränktheitsaussagen fur den d″ Operator*, (To appear).

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK, NEW YORK 10012