IMBEDDINGS, IMMERSIONS, AND COBORDISM OF DIFFERENTIABLE MANIFOLDS

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1. **Introduction.** The problem of imbedding a closed differentiable manifold M^n in a euclidean space can be weakened through the notion of (modulo 2) cobordism as follows. Is M^n cobordant to a submanifold of R^{n+k}? In this context we can prove an analogue, with improved dimensions, of H. Whitney's theorems [11], [12]. Let $\alpha(n)$ denote the number of ones in the binary expansion of n, and let $n>1$.

THEOREM A. *Any M^n is cobordant to a manifold N^n that imbeds in $R^{2n-\alpha(n)+1}$ and immerses in $R^{2n-\alpha(n)}$.*

For $n\neq 3$ this result is best possible as the examples below show. In some cases we can say more if certain Stiefel-Whitney numbers of M^n are zero. Allow the empty set as a representative of the zero cobordism class. (Thus Theorem A holds for all n.)

THEOREM B. (i) *If n is even ($n\neq 6$) and if $\bar{w}_{\alpha(n)}\cdot\bar{w}_{n-\alpha(n)}(M^n)=0$ then M^n is cobordant to a manifold N^n that imbeds in $R^{2n-\alpha(n)}$ and immerses in $R^{2n-\alpha(n)-1}$.*

(ii) *If $n=2^k$ or 2^k+1 and if $\bar{w}_i\cdot\bar{w}_{n-i}(M^n)=0$ for $0\leqq i\leqq s\leqq 3$ then M^n is cobordant to a manifold N^n that imbeds in R^{2n-s} and immerses in R^{2n-s-1}.*

Let $\mathfrak{N}_*$ denote the modulo 2 cobordism ring, and let $MO(k)$ denote the Thom complex for $O(k)$. There are homomorphisms

$$\Phi(n, k):\pi_{n+k}(MO(k))\rightarrow\mathfrak{N}_n \quad\text{and}\quad \Psi(n, k, N):\pi_{n+k+N}(S^N MO(k))\rightarrow\mathfrak{N}_n.$$

The image of $\Phi(n, k)$ is the set of cobordism classes that can be represented by submanifolds of R^{n+k} and hence coker $\Phi(n, k)=0$ if $k>n-\alpha(n)$ by Theorem A. The image of $\Psi(n, k, N)$ ($N\gg k$) is the set of cobordism classes that can be represented by manifolds which immerse in R^{n+k} (see R. Wells [10]) and hence coker $\Psi(n, k, N)=0$ if $k\geqq n-\alpha(n)$, $N\gg k$.

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Real projective n-space P^n ($n=2^k+1$, $k>1$) is known not to imbed in R^{2n-2} (see J. Levine [2]) but is cobordant to S^n which does. Complex projective n-space CP^n ($n=2^k$, $k>1$) does not immerse in R^{4n-2} (see J. Levine [3]) but is cobordant to $P^n \times P^n$ which does. Hence in Theorem B it is sometimes necessary to have $M^n \neq N^n$. However we know of no manifold M^n that does not imbed in $R^{2n-\alpha(n)+1}$ and immerse in $R^{2n-\alpha(n)}$.

2. **Decomposables in $\mathfrak{N}_*$.** The main theorems are proved by imbedding and immersing manifolds constructed from real projective spaces until we have enough to form a basis of $\mathfrak{N}_*$. We illustrate the method by outlining the proof of Theorem A.

PROPOSITION 2.1. *Suppose for each $n \neq 2^k-1$ there is a manifold V^n whose cobordism class $[V^n]$ is an indecomposable element of $\mathfrak{N}_*$ and which imbeds in $R^{2n-\alpha(n)+1}$ and immerses in $R^{2n-\alpha(n)}$. Then Theorem* A *holds.*

PROOF. According to R. Thom [9] the cobordism classes $[V^n]$ generate the ring $\mathfrak{N}_*$. Given a product $M^n = \prod V^j$ we can use the product immersion to immerse M^n in $(\sum(2j-\alpha(j)))$-space. Because $\alpha(i+j) \leqq \alpha(i)+\alpha(j)$ we have actually immersed M^n in $(2n-\alpha(n))$-space or better. The product imbedding is not good enough, so to imbed M^n in $(2n-\alpha(n)+1)$-space we use inductively the following well-known result. (For a three line proof see [7].)

LEMMA 2.2. *If M^m imbeds in R^s, N^n immerses in R^t, and $s+t>2n$ (which is true if $m \geqq n$) then $M^m \times N^n$ imbeds in R^{s+t}.*

Any M^n is cobordant to a disjoint union of products of the V^j and we can imbed and immerse this disjoint union in the obvious way, thus proving Theorem A.

3. **Construction of indecomposables.** Let n be even and let $n=r_1+\cdots+r_k$ $(2 \leqq r_1 < \cdots < r_k)$ be the binary expansion of n as a sum of distinct powers of 2. Thus $\alpha(n)=k$. Let $V^n=P^n$ if $k=1$ and for $k>1$ let V^n be a submanifold of $K^{n+1}=P^{r_k+1} \times \prod_{i=1}^{k-1} P^{r_i}$ dual to $\alpha_1+\cdots+\alpha_k \in H^1(K^{n+1}; Z_2)$ where α_i generates the modulo 2 cohomology ring of the ith factor.

PROPOSITION 3.1. *$[V^n]$ is an indecomposable element of $\mathfrak{N}_*$ and V^n satisfies the conditions of Proposition* 2.1.

PROOF. The first part follows from a computation of the total Stiefel-Whitney class $w(V^n)$ and from standard arguments using elementary symmetric functions (see R. E. Stong [8, p. 79]). The second part is based on an immersion of P^n ($n=2^s+1$) in R^{2n-3} due to B. J. Sanderson [6]. Whitney's results (M^n imbeds in R^{2n} and immerses in R^{2n-1}) and the product immersion or inductive use of Lemma 2.2 finish the proof.

REMARK 3.2. $M^n=\prod_{i=1}^k P^{r_i}$ has $\bar{w}_k\cdot\bar{w}_{n-k}\neq 0$ and hence furnishes a counterexample to improving Theorem A when n is even.

The above construction of even dimensional generators was inspired by the work of J. Milnor [5] and the following is a modification of A. Dold's construction of odd dimensional generators of $\mathfrak{N}_*$ [1]. Given a positive integer m and a topological space X form $P(m, X)$ from $S^m\times X\times X$ by identifying (u, x, y) with $(-u, y, x)$.

PROPOSITION 3.3. *$P(m, M^n)$ is an $(m+2n)$-manifold and represents an indecomposable element of $\mathfrak{N}_*$ if and only if $[M^n]$ is indecomposable and the binomial coefficient* $\binom{m+n-1}{n}\equiv 1 \pmod 2$.

A map $X\to Y$ induces a map $P(m, X)\to P(m, Y)$ and differentiable imbeddings and immersions are preserved by this functor. Also $P(m, R^s)$ is the total space $E(s\gamma_m\oplus s\epsilon)$ where γ_m, ϵ are respectively the canonical line bundle and the trivial line bundle over P^m. Thus we have proved

PROPOSITION 3.4. *If M^n imbeds (immerses) in R^s and $E(s\gamma_m\oplus s\epsilon)$ imbeds (immerses) in R^t then $P(m, M^n)$ imbeds (immerses) in R^t.*

Now let n be odd, $n\neq 2^k-1$. We can write uniquely $n=2^r(2s+1)-1=2^r-1+2^{r+1}s$ ($r>0$, $s>0$). Let $a=2^r-1$, $b=2^rs$ and $V^n=P(a, V^b)$.

PROPOSITION 3.5. *V^n satisfies Proposition* 2.1.

PROOF. By Propositions 3.1 and 3.3, $[V^n]$ is indecomposable. Using the imbedding and immersing part of Proposition 3.1 we can apply Proposition 3.4 to reduce the proof to imbedding and immersing certain sums of line bundles over P^a. Now the work of M. Mahowald and R. Milgram [4, Lemma 1.5] gives the required result.

REMARK 3.6. Using the notation of the beginning of this section let $M^{n+1}=P(1, \frac{1}{2}r_k)\times\prod_{i=1}^{k-1}P^{r_i}$. If $n>2$ then $\bar{w}_{k+1}\cdot\bar{w}_{n-k-1}(M^n)\neq 0$ so M^{n+1} serves as a counterexample to improving Theorem A.

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