ON BOUNDARY CONDITIONS FOR SYMMETRIC SUBMARKOVIAN RESOLVENTS

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1. **Introduction.** In a recent paper [4], M. Fukushima has established a one-to-one correspondence between symmetric markovian semigroups which satisfy the heat equation on a bounded domain D in Euclidean n-space and certain Dirichlet spaces on the Martin boundary of that domain. In this note we give an extension of his result to a much more general context.

Fukushima considers semigroups with resolvent kernels of the form

$$G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$$

where G_α^0 is the "absorbing barrier" or minimal resolvent for Brownian motion on D and $R_\alpha(x, y)$, defined for x and y in D, is a nonnegative, symmetric "α-harmonic" term, i.e. R_α satisfies the equation $\alpha R_\alpha - (1/2)\Delta R_\alpha = 0$ in D as a function of x for fixed y. Also, it is assumed that $\alpha G_\alpha 1 = 1$ in D. We start with a given nonnegative symmetric resolvent G_α^0 which is submarkovian, i.e. $\alpha G_\alpha^0 1 \leqq 1$, and then consider resolvents $G_\alpha \geqq G_\alpha^0$ which are symmetric and submarkovian. The Laplacian operator which plays a central role in Fukushima's work is here replaced by a much more general type of operator A which may not even be a local operator. The main results will be found in Theorems 1–3. Our method of proof is different from that of Fukushima. The details will be published elsewhere.

2. **Preliminaries.** Let (X, dx) be a sigma finite measure space and let $(\, , \,)_X$ or $(\, , \,)_{dx,X}$ denote the standard inner product on $L^2(X)$, the Hilbert space of real-valued square integrable functions on X.

2.1. DEFINITION. A symmetric submarkovian resolvent on $L^2(X)$ is a family $\{G_\alpha, \alpha > 0\}$ of bounded linear operators on $L^2(X)$ such that

2.1.1. $G_\alpha f \geqq 0$ a.e. whenever $f \geqq 0$ a.e. and $\alpha G_\alpha 1 \leqq 1$ a.e.

2.1.2. $G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta$.

2.2. DEFINITION. The measurable function g is a normalized contraction of the measurable function f if $|g(x)| \leqq |f(x)|$ and $|g(x) - g(y)| \leqq |f(x) - f(y)|$ for all x, y in X.

2.3. DEFINITION. A Dirichlet space relative to $L^2(X)$ is a pair $(F, \mathcal{E})$ where

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2.3.1. F is a linear (but not necessarily closed) subset of $L^2(X)$ and $\mathcal{E}$ is a positive semidefinite symmetric bilinear form on F.

2.3.2. For each $\alpha>0$ the linear space F is a Hilbert space relative to the inner product

$$\mathcal{E}_\alpha(f, g) = \mathcal{E}(f, g) + \alpha(f, g)_X.$$

2.3.3. If $f\in F$ and if g is a normalized contraction of f, then $g\in F$ and $\mathcal{E}(g, g) \leqq \mathcal{E}(f, f)$.

The connection between Dirichlet spaces and resolvents may be summed up as follows. Condition 2.1.2 guarantees that the G_α form a commuting family of bounded, symmetric operators in $L^2(X)$ and so the spectral theory can be applied to establish the existence of a negative definite operator A, the so-called generator of the resolvent family $\{G_\alpha, \alpha>0\}$, which is selfadjoint as an operator in the Hilbert space $\overline{R}$, the closure of the common range R of the G_α. Also $(\alpha-A)G_\alpha f =f$ for $f\in\overline{R}$, and $G_\alpha(\alpha-A)f=f$ for $f\in R$. Then F is just the domain of the unique positive square root $\sqrt{(-A)}$ of A and $\mathcal{E}(f, f) = (\sqrt{(-A)}f, \sqrt{(-A)}f)_X$ for $f\in F$. It follows easily that

$$\mathcal{E}(f,f) = \operatorname{Lim} \alpha(f - \alpha G_\alpha f, f)_X \qquad (\alpha \uparrow \infty). \tag{2.1}$$

The submarkovian property of the operators G_α is equivalent to the contraction property for the Dirichlet space $(F, \mathcal{E})$.

3. **The general set-up.** Let (D, dx) and $(M, d\zeta)$ be sigma finite measure spaces with the possibility that $d\zeta$ vanishes identically. Let $\Omega=D\cup M$. We suppose given once and for all a symmetric submarkovian resolvent $\{G_\alpha^0:\alpha>0\}$ on $L^2(D)$ which has been regularized in the following sense. Measures $G_\alpha^0(x, d\cdot)$ on D have been selected so that a representative for $G_\alpha^0 f$ is defined by the action of these measures, viz.

$$G_\alpha^0 f(x) = \int_D G_\alpha^0(x, dy)f(y)$$

and so with this choice of a representative the conditions 2.1.1 and 2.1.2 hold identically on D whenever they make sense. We assume further that 1 is excessive (that is, $\alpha G_\alpha^0 1$ increases to 1 identically as α increases to ∞), and we define h_a, the active part of 1, and h_p, the passive part of 1, by

$$h_p = \operatorname*{Lim}_{\alpha\downarrow 0} \alpha G_\alpha^0 1 \qquad h_a = 1 - h_p.$$

(See [3, §7] where the significance of h_a and h_p is discussed.) Our basic hypothesis is that h_a can be represented as

$$h_a(x) = \int_M K(x, \xi) d\mu(\xi) \tag{3.1}$$

where $d\mu$ is a bounded measure on M such that $d\zeta$ is absolutely continuous with respect to $d\mu$ and where $K(x, \xi)$ is a positive kernel defined and jointly measurable on $D \times M$ and satisfying:

3.4.1. For each ξ in M the function $K(\cdot, \xi)$ is excessive relative to the G^0_α.

3.4.2. For each x in D the function $K(x, \cdot)$ is bounded and bounded away from 0 on M.

3.4.3. If $\phi \in L^1(d\mu)$ and if both $\int_M K(\cdot, \xi)\phi(\xi)d\mu(\xi) = 0$ (a.e. dx) on D and $\phi = 0$ (a.e. $d\zeta$) on M, then $\phi = 0$ (a.e. $d\mu$).

In addition we assume

3.4.4. There exists a nonzero excessive function r belonging to $L^2(D)$. (Since 1 is excessive, this assumption is superfluous when D has finite dx measure.)

For $\alpha > 0$ the auxiliary kernel $K_\alpha(x, \xi)$ on $D \times M$ is defined by

$$K_\alpha(x, \xi) = K(x, \xi) - \alpha \int_D G^0_\alpha(x, dy) K(y, \xi).$$

For $\alpha \geqq 0$ and for ϕ in $L^1(d\mu)$, the function $H_\alpha\phi$ on Ω is defined by

$$\begin{aligned} H_\alpha\phi(x) &= \int_M K_\alpha(x, \xi)\phi(\xi)d\mu(\xi) \qquad x \in D, \\ &= \phi(x) \qquad\qquad x \in M. \end{aligned} \tag{3.2}$$

For $\alpha \geqq 0$ and for f on Ω we define

$$\hat{H}_\alpha f(\xi) = \int_D f(x) K_\alpha(x, \xi) dx + \frac{d\zeta}{d\mu}(\xi) f(\xi) \qquad (\xi \in M).$$

For $\xi, \eta \in M$, for ϕ, ψ defined on M, and for $0 \leqq \beta \leqq \alpha$ we introduce

$$U^0_{\beta,\alpha}(\xi, \eta) = (\alpha - \beta) \int_D dx K_\alpha(x, \xi) K_\alpha(x, \eta),$$

$$U_{\beta,\alpha}\phi(\xi) = \int_M U^0_{\beta,\alpha}(\xi, \eta)\phi(\eta)d\mu(\eta) + (\alpha - \beta)\frac{d\zeta}{d\mu}(\xi)\phi(\xi),$$

$$U^0_{\beta,\alpha}\langle\phi, \psi\rangle = \int_M d\mu(\eta) \int_M d\mu(\xi) \{\phi(\xi) - \phi(\eta)\}\{\psi(\xi) - \psi(\eta)\} U^0_{\beta,\alpha}(\xi, \eta),$$

$$U_{\beta,\alpha}(\phi, \psi) = \int_M d\mu(\xi)\phi(\xi) U_{\beta,\alpha}\psi(\xi),$$

$$p(\xi) = \hat{H}_1 h_p(\xi).$$

It is easy to check that

$$U_{\beta,\gamma}(\phi, \psi) = U_{\beta,\alpha}(\phi, \psi) + U_{\alpha,\gamma}(\phi, \psi), \qquad 0 \leqq \beta < \alpha < \gamma,$$

$$\overset{0}{U}_{\beta,\gamma}\langle\phi, \psi\rangle = \overset{0}{U}_{\beta,\alpha}\langle\phi, \psi\rangle + \overset{0}{U}_{\alpha,\gamma}\langle\phi, \psi\rangle, \qquad 0 \leqq \beta < \alpha < \gamma.$$

The last two relations permit us to introduce

$$\overset{0}{U}_{\beta,\infty}(\xi, \eta) = \operatorname{Lim} \overset{0}{U}_{\beta,\alpha}(\xi, \eta) \qquad (\alpha \uparrow \infty),$$

$$\overset{0}{U}_{\beta,\infty}\langle\phi, \psi\rangle = \operatorname{Lim} \overset{0}{U}_{\beta,\alpha}\langle\phi, \psi\rangle \qquad (\alpha \uparrow \infty).$$

Finally, we introduce the measure $d\mu^1(\xi) = \hat{H}_1 h_\alpha(\xi) d\mu(\xi)$ on M and the pair $(H^R, \mathcal{E}^R)$ viz.

$$H^R = \{\phi \in L^2(d\mu^1) : \mathcal{E}^R(\phi, \phi) < \infty\},$$

$$\mathcal{E}^R(\phi, \psi) = \tfrac{1}{2}\overset{0}{U}_{0,\infty}\langle\phi, \psi\rangle + \int_M d\mu(\xi) p(\xi)\phi(\xi)\psi(\xi).$$

It is easily checked that $(H^R, \mathcal{E}^R)$ is a Dirichlet space relative to $L^2(d\mu^1, M)$. (The significance of the superscript R will be explained after Theorem 2 in §4.)

4. **The main result.** A submarkovian resolvent $\{G_\alpha : \alpha > 0\}$, symmetric relative to $dx + d\zeta$ on Ω, will be called admissible if

4.1.1. For $f \geqq 0$ on Ω, a version for $G_\alpha f$ has been selected which is defined identically on D and (a.e. $d\mu$) on M and such that 2.1.1 and 2.1.2 are valid identically on D and (a.e. $d\mu$) on M.

4.1.2. $G_\alpha f = G_\alpha^0 + H_\alpha G_\alpha f$ identically on D for $\alpha > 0$.

4.2 CONVENTION. The resolvent $\{G_\alpha^0 : \alpha > 0\}$ is regarded as acting on functions on Ω in the obvious way: $G_\alpha^0 f = 0$ on M and $G_\alpha^0 f = G_\alpha^0 f'$ on D where f' is the restriction of f to D. Then G_α^0 is an admissible resolvent, and 4.1.2 is now true not only on D but on all of Ω.

Given any admissible resolvent $\{G_\alpha : \alpha > 0\}$ we denote the associated generator by A and the associated Dirichlet space norms by $\mathcal{E}$ and $\mathcal{E}_\alpha$.

THEOREM 1. *If $\{G_\alpha : \alpha > 0\}$ is an admissible resolvent on Ω, then there is a unique pair $(H^M, \mathcal{E}^M)$ where*

(i) *H^M is a linear subset of H^R which is stable with respect to normalized contractions.*

(ii) *$\mathcal{E}^M$ is a bilinear form on H^M which dominates $\mathcal{E}^R$ in the following sense: if ϕ is in H^M and if ψ is a normalized contraction of ϕ, then*

$$(4.1) \qquad 0 \leqq \mathcal{E}^M(\psi, \psi) - \mathcal{E}^R(\psi, \psi) \leqq \mathcal{E}^M(\phi, \phi) - \mathcal{E}^R(\phi, \phi).$$

(iii) *The pair $(H^M, \mathcal{E}^M)$ is a Dirichlet space relative to $L^2(d\mu^1)$.*

(iv) *For each* $\alpha>0$ *the operator* H_α *maps* H^M *into the domain of* $\sqrt{(-A)}$ *and for* ϕ *in* H^M

$$\mathcal{E}_\alpha(H_\alpha\phi, H_\alpha\phi) = \mathcal{E}^M(\phi, \phi) + U_{0,\alpha}(\phi, \phi).$$

Moreover, domain $\sqrt{(-A^0)}$ *and the closure of* $H_\alpha(H^M)$ *are complementary orthogonal subspaces in domain* $\sqrt{(-A)}$ *relative to the inner product* $\mathcal{E}_\alpha$.

Conversely, to every pair $(H^M, \mathcal{E}^M)$ *satisfying* (i), (ii), *and* (iii) *there is associated a unique admissible resolvent such that* (iv) *is valid.*

Theorem 1 yields a complete characterization of admissible resolvents $\{G_\alpha:\alpha>0\}$ by means of Dirichlet spaces $(H^M, \mathcal{E}^M)$ on M. The next theorem gives additional information on the connection between $(H^M, \mathcal{E}^M)$ and $\{G_\alpha:\alpha>0\}$.

THEOREM 2. *Let* $\{G_\alpha:\alpha>0\}$ *and* $(H^M, \mathcal{E}^M)$ *be as in Theorem* 1 *and for* $\alpha>0$, *and for* ϕ, ψ *in* H^M *let*

$$\mathcal{E}_\alpha^M(\phi, \psi) = \mathcal{E}^M(\phi, \psi) + U_{0,\alpha}(\phi, \psi).$$

Then

(i) *For each* $\alpha>0$ *the pair* $(H^M, \mathcal{E}_\alpha^M)$ *is a Dirichlet space relative to* $L^2(d\mu)$.

(ii) *If* $\{\tilde{R}_{\alpha;\lambda}:\lambda>0\}$ *is the resolvent on* $L^2(d\mu)$ *corresponding to the Dirichlet space* $(H^M, \mathcal{E}_\alpha^M)$, *then* $\tilde{R}_\alpha=\operatorname{Lim} \tilde{R}_{\alpha;\lambda}$ $(\lambda\downarrow 0)$ *exists in the strong operator topology on* $L^2(d\mu)$, *and for* f *on* Ω

$$G_\alpha f = \overset{0}{G_\alpha} f + H_\alpha \tilde{R}_\alpha \hat{H}_\alpha f.$$

REMARK. The resolvent $\{G_\alpha^0:\alpha>0\}$ corresponds to the absorbing barrier resolvent in [4] and to the minimal resolvent in [1], [2], and [3]. The Dirichlet space $(H^R, \mathcal{E}^R)$ is the analogue of the boundary Dirichlet space associate in [4] with the reflecting barrier process (hence the superscript R). Just as in [4] it will turn out that the relevant Dirichlet spaces on the boundary are just those contained in $(H^R, \mathcal{E}^R)$ in an appropriate sense.

To formulate the results in terms of boundary conditions we introduce the following two operators:

4.3. F defined on Ω is in the domain of the local 1-generator A_1 if F is in the domain of $\sqrt{(-A^R)}$ and if there exists f in $L^2(D)$ such that $\mathcal{E}_1^R(F, g) = -\int_D dx\, f(x)g(x)$ for g in the domain of $\sqrt{(-A^0)}$. In this case $A_1F=f$. (Here A^0 is the absorbing barrier generator and A^R the reflecting barrier generator. It follows from Theorem 1(iv) that

$$\text{domain } \sqrt{(-A^0)} \subset \text{domain } \sqrt{(-A)} \subset \text{domain } \sqrt{(-A^R)}$$

for A the generator of an admissible resolvent.)

4.4. F defined on Ω is in the domain of A_1^+, the extended 1-generator if F is in the domain of the local 1-generator A_1, if the boundary value γF of F belongs to H^M and if there exists ϕ in $L^2(M, d\zeta)$ such that

$$\mathcal{E}_1^M(\gamma F, \psi) + \int_M d\mu(\xi)\psi(\xi)\hat{H}_1 A_1 F(\xi) = \int_M d\zeta(\xi)\psi(\xi)\phi(\xi)$$

for all ψ in H^M. In this case

$$\begin{aligned} A_1^+ F &= A_1 F \quad \text{on } D, \\ &= \phi \quad \text{on } M. \end{aligned}$$

(In the course of proving Theorem 1 we show that the boundary value γF of F is well defined as an element of $L^1(M, d\mu)$ for F in domain $\sqrt{(-A^R)}$ and in particular for F in domain A_1.) Then we have

THEOREM 3. *Let A be the generator of the admissible resolvent* $\{G_\alpha : \alpha > 0\}$.

(i) *If $d\zeta = 0$, then F is in the domain of A if and only if F is in the domain of the local 1-generator A_1 and*

$$\mathcal{E}_1^M(\gamma F, \psi) + \int_M d\mu(\xi)\psi(\xi)\hat{H}_1 A_1 F(\xi) = 0$$

for all ψ in H^M. In this case $AF = A_1F + F$.

(ii) *If $d\zeta \neq 0$, then F is in the domain of A if and only if F is in the domain of the extended 1-generator A_1^+ and then $AF = A_1^+F + F$.*

REMARK. Concerning the interpretation of the Dirichlet space $(H^M, \mathcal{E}^M)$ in terms of "Markov processes on the boundary" we refer the reader to [5] and [6]. It remains to be established to what extent or in what sense this interpretation is valid in the present context.

REFERENCES

1. J. Elliott, *Dirichlet spaces associated with integro-differential operators*. I, Illinois J. Math. **9** (1965), 87–98. MR **30** #5159.

2. ———, *Lateral conditions for semigroups involving mappings in L^p*. I, J. Math. Anal. Appl. **25** (1969), 388–410. MR **39** #3350.

3. W. Feller, *On boundaries and lateral conditions for the Kolmogorov differential equations*, Ann. of Math. (2) **65** (1957), 527–570. MR **19**, 892.

4. M. Fukushima, *On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities*, J. Math. Soc. Japan **21** (1969), 58–93. MR **38** #5291.

5. K. Sato, *A decomposition of Markov processes*, J. Math. Soc. Japan **17** (1965), 269–293. MR **31** #6284.

6. K. Sato and T. Ueno, *Multi-dimensional diffusion and the Markov process on the boundary*, J. Math. Kyoto Univ. **4** (1964/65), 529–605. MR **33** #6702.

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