ON STAR-INVARIANT SUBSPACES

BY P. R. AHERN[1] AND D. N. CLARK[2]

Communicated by Gian-Carlo Rota, September 22, 1969

Let H^2 denote the usual Hardy class of functions holomorphic in the unit disk. Let M denote a closed, invariant subspace of H^2. The theory of such subspaces is well known; every such M has the form $M=\phi H^2$, where $\phi\in H^2$ is an inner function, $\phi=Bs\Delta$, with

$$B(z) = \prod_{\nu=1}^{\infty} \frac{\bar{a}_\nu}{|a_\nu|}\frac{z-a_\nu}{1-\bar{a}_\nu z}, \qquad s(z) = \exp\left\{-\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\sigma_s(\theta)\right\},$$

$$\Delta(z) = \exp\left\{-\sum_{\nu=1}^{\infty} r_\nu \frac{e^{i\theta_\nu}+z}{e^{i\theta_\nu}-z}\right\}$$

where $\{a_\nu\}$ is a Blaschke sequence ($\bar{a}_\nu/|a_\nu|\equiv 1$ if $a_\nu=0$), σ_s is a finite, positive, continuous, singular measure, and $r_\nu\geqq 0$, $\sum r_\nu<\infty$.

Less is known about the "star-invariant" subspaces $M^\perp=H^2\ominus M$. In this announcement, we outline some results we have obtained recently concerning the subspace $M^\perp$. Full details and proofs will appear elsewhere.

1. **A unitary operator.** In our first theorem, we represent $M^\perp$ unitarily as the sum of the spaces $L^2(d\sigma_B)$, $L^2(d\sigma_s)$ and $L^2(d\sigma_\Delta)$. Here σ_B is the measure on the positive integers which assigns a mass $1-|a_k|$ to the integer k; σ_Δ is the measure on $[0,\infty]$ which is r_k times Lebesgue measure on the interval $[k-1,k]$; and σ_s is the measure associated with s above.

In the special case $\phi=B$, our unitary operator $V_B: L^2(d\sigma_B)\rightarrow(BH^2)^\perp$ is given by

$$V_B(\{c_n\})(z) = \sum_{n=1}^{\infty} c_n(1+|a_n|)^{1/2}B_n(z)(1-\bar{a}_n z)^{-1}(1-|a_n|).$$

Here B_n is the partial product of B with zeros $a_1,\cdots,a_{n-1}$. The fact that V_B is unitary is a consequence of the simple and well-known fact that the functions $h_n(z)=(1-|a_n|^2)^{1/2}B_n(z)/(1-\bar{a}_n z)$ form an orthonormal basis of $(BH^2)^\perp$.

AMS Subject Classifications. Primary 4630; Secondary 3065, 3067, 4725.

Key Words and Phrases. Invariant subspace, inner function, L^2 space, shift operator, restricted shift operator.

[1] Supported by NSF Grant GP-6764.

[2] Supported by NSF Grant GP-9658.

If $\phi=\Delta$, $V_\Delta: L^2(d\sigma_\Delta)\to(\Delta H^2)^\perp$ is defined by

$$(V_\Delta c)(z) = \int_0^\infty c(\lambda)\sqrt{2\Delta_\lambda(z)}(1 - e^{-\theta N+1}z)^{-1}d\sigma_\Delta(\lambda)$$

where

$$\Delta_\lambda(z) = \exp\left\{-\sum_{j=1}^{N} r_j \frac{e^{i\theta_j}+z}{e^{i\theta_j}-z} - (\lambda - N)r_{N+1}\frac{e^{i\theta N+1}+z}{e^{i\theta N+1}-z}\right\}$$

and N is the integral part of λ. If $r_\nu=0$, $\nu\neq 1$ and $\theta_1=0$, V_Δ is the unitary operator defined by Sarason, in [**5**].

Finally, if $\phi=s$, we set

$$s_\lambda(z) = \exp\left\{-\int_0^\lambda \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\sigma_s(\theta)\right\}$$

and let $V_s: L^2(d\sigma_s)\to(sH^2)^\perp$ be defined by

$$(V_s c(\lambda))(z) = \int_0^{2\pi} c(\lambda)\sqrt{2s_\lambda(z)}(1 - e^{-i\lambda}z)^{-1}d\sigma_s(\lambda).$$

Our three special cases may now be combined in

THEOREM 1. *The operator*

$$V: L^2(d\sigma_B)\times L^2(d\sigma_s)\times L^2(d\sigma_\Delta)\to(Bs\Delta H^2)^\perp$$

defined by

$$V(c_B, c_s, c_\Delta) = V_B c_B + BV_s c_s + BsV_\Delta c_\Delta$$

is an isometry onto $(Bs\Delta H^2)^\perp$.

2. **The restricted shift.** In this section we consider the restricted shift operator T on $(\phi H^2)^\perp$, defined by

$$Tf = Pzf \qquad f\in(\phi H^2)^\perp$$

where P is the projection onto $(\phi H^2)^\perp$. We want to find the form of the operator V^*TV, unitarily equivalent to T under V.

Again we begin with the special cases $\phi=B$, s, Δ. We define K_B, K_s and K_Δ as the integral operators on $L^2(d\sigma_B)$, $L^2(d\sigma_s)$ and $L^2(d\sigma_\Delta)$ given by

$$K_B c(n) = \sum_{j=1}^{n} c(j)B_n(0)/B_j(0)(1+|a_j|)|a_j|^{-2}(1-|a_j|)$$

for $\phi=B$ and by

$$K_\phi c(\lambda) = 2\int_0^\lambda c(t)\phi_\lambda(0)/\phi_t(0)d\sigma_\phi$$

for $\phi = s, \Delta$. We define multiplication operators M_B, M_s and M_Δ by

$$(M_B c)(n) = a_n c(n) \qquad (M_s c)(\lambda) = e^{i\lambda}c(\lambda)$$

and

$$(M_\Delta c)(\lambda) = e^{i\theta_{N+1}}c(\lambda) \quad \text{on } N \leqq \lambda < N+1.$$

Our result for the special cases $\phi = B, s, \Delta$ is that

$$V_\phi^* T V_\phi = (I - K_\phi)M_\phi \equiv A_\phi.$$

Combining these results, we have

THEOREM 2. *V^*TV is an operator A on $L^2(d\sigma_B)\times L^2(d\sigma_s)\times L^2(d\sigma_\Delta)$ given by*

$$A(c_B,\ c_s,\ c_\Delta) = (A_B c_B,\ A_s c_s + \alpha_B(c_B)k_s,\ A_\Delta c_\Delta + \alpha_B(c_B)s(0)k_\Delta + \alpha_s(c_s)k_\Delta)$$

where k_ϕ is V_ϕ^* of the projection of 1 on $(\phi H^2)\perp$ for $\phi = s, \Delta$, and α_B, α_s are functionals.

3. **Applications.** Theorem 2 has applications to spectral properties of certain functions of T, i.e. to operators T_u defined by

$$T_u f = Puf \qquad f \in M^\perp.$$

In fact, V^*TV is the sum of a multiplication operator M:

$$M(c_B, c_s, c_\Delta) = (M_B c_B, M_s c_s, M_\Delta c_\Delta)$$

and an operator K which is easily seen to be of Hilbert-Schmidt class:

$$K(c_B,\ c_s,\ c_\Delta) = (K_B c_B,\ K_s c_s + \alpha_B(c_B)k_s,\ K_\Delta c_\Delta + \alpha_B(c_B)s(0)k_\Delta + \alpha_s(c_s)k_\Delta).$$

Thus, if u is continuous in $|z| \leqq 1$, $T_u = u(M) + K'$, where K' is compact. From this it is easy to determine the spectrum of T (cf. Moeller [**4**]) and of certain functions of T (cf. Foiaş-Mlak [**3**]). Certain other facts about T are consequences of Theorem 2, for example

THEOREM 3. *If u is continuous in $|z| \leqq 1$, then T_u is compact if and only if $u \equiv 0$ on* supp $\phi \cap \{|z| = 1\}$.

Here supp ϕ denotes the closure of the union of the set of zeros of B, the support of σ_s and the numbers $e^{i\theta_j}$, $j = 1, 2, \cdots$.

THEOREM 4. *If $2 \leqq p < \infty$ and u is a trigonometric polynomial, then $T_u \in c_p$ if and only if*

(i) $u \equiv 0$ *on* supp $s\Delta$, *and*

(ii) $\{u(a_k)\} \in l^p$.

In addition, Theorems 1 and 2 have applications to a problem we studied in [1] and to give the following affirmative answer to a question raised in [2] by Douglas, Shapiro and Shields.

THEOREM 5. *Let Y denote the shift on all of H^2:*

$$Yf = zf \qquad f \in H^2.$$

Then, if ϕ is any inner function not of the form $\phi = e^{in\theta}$, the set $\{Y^*\psi\}$, for ψ a divisor of ϕ $(\psi \neq \phi)$, spans $(\phi H^2)^\perp$.

We close by noting that some close analogs to Theorems 1 and 2 above were discovered independently by T. L. Kriete, III.

REFERENCES

1. P. R. Ahern and D. N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. (to appear).
2. R. G. Douglas, H. S. Shapiro and A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, (preprint).
3. C. Foiaş and W. Mlak, *The extended spectrum of completely non-unitary contractions and the spectral mapping theorem*, Studia Math. **26** (1966), 239–245. MR **34** #610.
4. J. W. Moeller, *On the spectra of some translation invariant spaces*, J. Math. Anal. Appl. **4** (1962), 276–296. MR **27** #588.
5. D. Sarason, *A remark on the Volterra operator*, J. Math. Anal. Appl. **12** (1965), 244–246. MR **33** #580.

UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706 AND
UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024