EXPRESSION FOR A FUNCTION IN TERMS OF ITS SPHERICAL MEANS

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Let $f(X)$ be a continuous function in R^n. The spherical means, SM, of f is defined as follows:

$$\mathrm{SM}[f; X, \rho] = \omega_n^{-1} \int_\alpha f(X + \rho_\alpha) d\omega_\alpha,$$

where $X = (x, x_2, x_3, \cdots, x_n)$ is the center of the sphere of radius ρ. α denotes a unit vector. When $\rho = x$, we write $\mathrm{SM}[f; X, x] = \mathrm{SM}^*f$. The main purpose of this paper is to derive an expression for a function $f(X)$, $X \in R_+^n$ (the open half-space with $x > 0$), in terms of SM $^*f$. For $(X, t) \in Q_+$ ($|t| < x$, $-\infty < x' < \infty$, $x' = (x_2, x_3, \cdots, x_n)$, $(x, x') \in R_+^n$, n odd$\geqq 3$) we define the paraboloidal means, PM, of f as follows:

$$\mathrm{PM}[f; X, t] = \omega_{n-1}^{-1}(x + t)^{2-n} \int_b^\infty dy \int_\alpha f(y, x' + R\alpha) R^{n-3} d\omega_\alpha,$$

where $b = (x-t)/2$, $Y = (y, y')$, $R = [(x+t)(2y-x+t)]^{1/2}$.

A function $f(X)$ is said to belong to the class C_ϵ in R_+^n, if f is continuous in R_+^n and $f(X) = O(|X|^{(1-n-2\epsilon)/2})$, $0 < \epsilon < 1$, for large $|X|$. We observe that PM $[f; X, t]$ exists, if $f \in C_\epsilon$. It is easily verified that if $f \in C_\epsilon$, then $\mathrm{SM}^*f \in C_\epsilon$. The well-known identity on iterated spherical means by John and Asgeiersson [3] states

$$(1) \qquad \int_\xi d\omega_\xi \int_\eta F(r\xi + s\eta) d\omega_\eta = 2\omega_{n-1} \int_{|r-s|}^{r+s} J\tau d\tau \int_\zeta F(\tau\zeta) d\omega_\zeta,$$

where $J = [((r+s)^2 - \tau^2)(\tau^2 - (r-s)^2)]^{(n-3)/2}(2rs)^{2-n}$.

THEOREM. *Let $f \in C_\epsilon$ in R_+^n (n odd$\geqq 3$), and let $W(X, t) = (x+t)^{n-2}\mathrm{PM}[\mathrm{SM}^*f; X, t]$. Then the following identity holds for $(X, t) \in Q_+$,*

$$(2) \quad t\mathrm{SM}[f; X, t] = M_1 D D_0^{n-3} W(X, t) + M_2 \sum_{i=1}^{(n-3)/2} a_i D_1^i t^{i+1} \mathrm{SM}[f; X, t],$$

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where

$$M_1 = (-1)^{(n-1)/2}\Gamma^{-1}(n-2),\ D = x\partial/\partial t + t\partial/\partial x,$$
$$D_0 = (x+t)^{-1}D,\ M_2 = -\Gamma^{-1}(n-2)2^{(3n-11)/2}\Gamma(k+1),\ k = (n-3)/2,$$
$$a_i = \Gamma(k+i)[2^{i-1}\Gamma(i)\Gamma(k-i+2)]^{-1},\ D_1 = D(x+t)^{-1}.$$

(The lengthy proof of the theorem which makes use of (1) will be submitted elsewhere.)

From (2) it follows that

$$f(X) = M_3 x^{-1}[D^2 D_0^{n-3} W(X, t)]_{t=0}, \tag{3}$$

where $M_3 = M_1[1 - M_2 \sum_{i=1}^{(n-3)/2} a_i\Gamma(i+2)]^{-1}$ for $n>3$, $M_3 = -1$ for $n=3$. (3) is an expression for $f(X)$, $X \in R^n_+$, in terms of the paraboloidal means of SM$^*f$.

LEMMA. *Suppose $f \in C_\epsilon$ and $x\ \partial f/\partial x \in C_\epsilon$. Let $W_0[f; X, t] = (x+t)^{n-2}\mathrm{PM}[f; X, t]$. Then we have*

$$DW_0[f; X, t] = -W_0[\delta f; X, t], \quad \textit{where } \delta = (\partial/\partial x)x. \tag{4}$$

PROOF. It is easily seen that $DR = yR_y$, $DR^{n-3}f(y, x'+R\alpha) = d/dy(yR^{n-3}f) - R^{n-3}\partial/\partial y(yf)$. Integrate this expression with respect to y to complete the proof.

Applying (4) to (3) and letting $J(X) = \mathrm{SM}^*f$, we obtain

$$f(X) = -(2\pi x)^{-1}\int_{x/2}^{\infty} dy \int_\alpha \delta^2 J(y, x' + R_0\alpha)\,d\omega_\alpha \qquad \text{for } n = 3,$$

$$f(X) = M_3\omega_{n-1}^{-1}x^{2-n}\sum_{i=1}^{n-1} b_i \int_{x/2}^{\infty} dy \int_\alpha R_0^{n-3}\delta^i J(y, x'R_0\alpha)\,d\omega_\alpha \qquad \text{for } n > 3,$$

where $R_0 = [x(2y-x)]^{1/2}$, the constants b_i are the coefficients in the expansion of $DD_1^{n-3}g(X)$, $g \in C^{n-2}$.

REMARK. It follows from (2) that if we can find $f(X)$ which satisfies the equation SM$^*f = J(X)$ for a given function J, then we will have a representation in Q_+ for the even-solutions of the Darboux equation $V_{tt} + (n-1)/t\,V_t - \Delta V = 0$ in terms of $J(X) = V(X, \pm x)$ (that is, if the equation SM$^*f = J$ can be inverted, then we have an expression for the even-solutions of the Darboux equation in Q_+ in terms of prescribed values on the characteristics C_+ ($x=t$, $x>0$) and C_- ($x=-t$, $x>0$)). The problem of inverting SM$^*f = J(X)$ in R^3 was studied by Chen [1], [2]. It was shown in [4] that inverting the equation SM$[f; X, |X|] = J(X)$ leads to a representation of the even-solutions of the Darboux equation in the exterior of the characteristic cones C_0 ($t = \pm |X|$, $X \in R^n$, n odd $\geqq 3$) in terms of prescribed values on C_0.

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